

Correlation functions and the algebraic Bethe ansatz in the AdS/CFT correspondence

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Abstract

Inverse scattering and the algebraic Bethe ansatz can be used to reduce the evaluation of form factors and correlation functions to the calculation of a product of Bethe states. In this article we develop a method to compute correlation functions of spin operators located at arbitrary sites of the spin chain. We will focus our analysis on the $SU(2)$ sector of $\mathcal{N} = 4$ supersymmetric Yang-Mills at weak-coupling. At one-loop we provide a systematic treatment of the apparent divergences arising from the algebra of the elements of the monodromy matrix of an homogeneous spin chain. Beyond one-loop the analysis can be extended through the map of the long-range Bethe ansatz to an inhomogeneous spin chain. We also show that a careful normalization of states in the spin chain requires choosing them as Zamolodchikov-Faddeev states.

1 Introduction

In an integrable model the spectrum and the eigenstates of the corresponding Hamiltonian and the scattering matrix can be entirely determined. Integrability however does not suffice to provide a complete and exact description of the correlation functions of the theory. An appealing approach to this open question comes from a combination of algebraic Bethe ansatz techniques with the solution to the quantum inverse scattering problem [1]. Finding a solution to the inverse scattering problem means to write the local operators of a quantum spin chain in terms of the elements of the monodromy matrix of the system. The algebra satisfied by the entries of the monodromy matrix can then be employed to reduce the computation of correlation functions to the scalar product of a Bethe eigenstate with some general reference states (see [2] for a review and references therein). The problem can be further reduced because scalar products of Bethe eigenstates with generic Bethe states can be expressed in terms of determinants [3, 4].

A natural place to apply this approach is the integrable system underlying the AdS/CFT correspondence. As in other systems, integrability in the AdS/CFT correspondence has lead to a very precise knowledge of the spectrum of local gauge invariant operators and to the derivation of an explicit form of the scattering matrix (see for instance [5]). However, as there is not yet an equivalent understanding of generic correlation functions, the method developed in [1] could probably be employed to shed some light on the spectrum of correlation functions in the correspondence. The algebraic Bethe ansatz and the solution to the inverse scattering problem were first used in [6] to evaluate three-point functions of scalar operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills as inner products of Bethe states, constructed out of the elements of the monodromy matrix. The structure constants of the theory could then expressed in terms of some elegant determinant expressions in [7]-[21].

The promising path started from the previous developments makes desirable a more detailed study of general correlation functions using algebraic Bethe ansatz techniques. In this article we will consider the case where the spin operators are located at non-adjacent sites. We will focus on the $SU(2)$ sector of $\mathcal{N} = 4$ supersymmetric Yang-Mills. At one-loop the dilatation operator in that sector reduces to the hamiltonian of an homogeneous Heisenberg chain [22]. At two and three-loops the spectrum of anomalous dimensions can be obtained from the Bethe ansatz equations of a long-range spin chain, with interactions beyond nearest-neighbours [23]. Our analysis will start with the one-loop homogeneous

Heisenberg chain, where some care will be needed in order to regularize the seeming divergent behavior of the commutation relations of the elements of the monodromy matrix. We will then extend the problem to the long-range Bethe ansatz recalling that it can be mapped to an inhomogeneous Heisenberg spin chain. Our results will however not include the contribution from the dressing phase factor, that needs to be included to match the spectrum of anomalous dimensions at four-loops and beyond.

The remaining part of the article is organized as follows. In section 2 we review the coordinate and the algebraic Bethe ansatz for the isotropic Heisenberg spin chain. We include the solution to the inverse scattering problem and a discussion on the normalization of states in each ansatz. Then in section 3 we will use the inverse scattering method to bring the calculation of form factors of spin operators to the scalar product of one Bethe state with an arbitrary vector. In section 4 we will face the problem of correlation functions involving two spin operators. This will require some care, because the commutation relations of the elements of the monodromy matrix for the homogeneous spin chain diverge. The method that we will develop to regularize these divergences will be the central part of this article. In section 5 we extend the analysis to the long-range Bethe ansatz. This can be done rather straightforwardly, recalling that the long-range spin chain can be mapped to an inhomogeneous short-range spin chain. In section 6 we conclude with several remarks together with a discussion on our results and the form factor program. We conclude the paper with three appendices that collect several additional results. In appendix A we find the recurrence relation of the most general correlation function involving two spin operators and two magnons. In appendix B we extend our analysis to spin chains with $SL(2)$ and $SU(1|1)$ symmetries. In appendix C we will consider correlation functions involving three magnons.

2 The Bethe ansatz

In this section we will review some relevant aspects of the coordinate and the algebraic Bethe ansatz for the spin 1/2 XXX Heisenberg spin chain. Together with an abridged presentation of the solution to the inverse scattering problem and the scalar product of Bethe states, we include a discussion on the different normalization of states in the coordinate and the algebraic Bethe ansatz, that will prove of crucial relevance when comparing computations along the paper.

2.1 The coordinate Bethe ansatz

We will focus our analysis on the $SU(2)$ sector of $\mathcal{N} = 4$ supersymmetric Yang-Mills. At one-loop the dilatation operator is the homogeneous spin 1/2 Heisenberg hamiltonian [22],

$$H = g^2 \sum_{n=1}^L (\mathbb{I}_{n,n+1} - \mathbb{P}_{n,n+1}) , \quad (2.1)$$

where L is the number of sites of the chain, $\mathbb{P}_{n,n+1}$ is the permutation operator acting at positions n and $n+1$, and the coupling constant is related to the 't Hooft coupling by

$$g^2 = \frac{\lambda}{8\pi^2} = \frac{g_{\text{YM}}^2 N}{8\pi^2} . \quad (2.2)$$

The basic statement behind the coordinate Bethe ansatz (CBA from now on) is the assumption that a generic state of N magnons,

$$|\Psi\rangle = \sum_{1 \leq n_1 < n_2 < \dots < n_N \leq L} \psi(n_1, \dots, n_N) |n_1, \dots, n_N\rangle , \quad (2.3)$$

which diagonalizes the Heisenberg spin chain, can be written as the weighted sum of all the possible free wave functions we can construct, that is,

$$\psi(n_1, \dots, n_N) = \sum_{\sigma \in \mathcal{P}_N} A(\sigma, \vec{p}) e^{i(p_1 \sigma(n_1) + p_2 \sigma(n_2) + \dots + p_N \sigma(n_N))} , \quad (2.4)$$

where \mathcal{P}_N is the permutation group of N elements and $A(\sigma, \vec{p})$ is a function that depends on the element of the permutation group and the momenta of each magnon, p_i . If we solve the Schrödinger equation $\hat{H} |\Psi\rangle = E |\Psi\rangle$ we obtain the dispersion relation

$$E = \sum_{j=1}^N \epsilon(p_j) , \quad \epsilon(p) = 4g^2 \sin^2 \left(\frac{p}{2} \right) . \quad (2.5)$$

The wave function can be written in terms of the S-matrix,

$$S(p, q) = \frac{\frac{1}{2} \cot \left(\frac{p}{2} \right) - \frac{1}{2} \cot \left(\frac{q}{2} \right) + i}{\frac{1}{2} \cot \left(\frac{p}{2} \right) - \frac{1}{2} \cot \left(\frac{q}{2} \right) - i} , \quad \text{with} \quad A(\mathbb{P}_{j,j+1} \sigma) = S(p_j + 1, p_j) A(\sigma) , \quad (2.6)$$

where we have assumed that $A(\mathbb{I}) = 1$. For instance, for a two-magnon state

$$\psi(n_1, n_2) = e^{i(p_1 n_1 + p_2 n_2)} + S(p_2, p_1) e^{i(p_2 n_1 + p_1 n_2)} , \quad (2.7)$$

which corresponds to an incoming wave $e^{i(p_1 n_1 + p_2 n_2)}$ for two magnons with respective momenta p_1 and p_2 , and an outgoing wave with the momenta exchanged and a relative coefficient given by the two-particle S-matrix. The calculation of the set of allowed momenta

p_i is performed using the periodicity of the spin chain, which imposes N quantization conditions for each of the momenta known as the Bethe ansatz equations,

$$e^{ip_j L} \prod_{k \neq j}^N S(p_k, p_j) = 1 . \quad (2.8)$$

The physical meaning of this equation is that if we carry one magnon with momentum p_j around the circle, the free propagation phase $p_j L$ plus the phase change due to the scattering with each of the other $N - 1$ magnons must give a trivial phase.

In the case of $\mathcal{N} = 4$ supersymmetric Yang-Mills there is an additional condition that needs to be imposed on physical states. Physical states correspond to single-trace operators. Cyclicity of the trace implies that they have to satisfy the zero momentum condition

$$\prod_{i=1}^N e^{ip_i} = 1 . \quad (2.9)$$

2.2 The algebraic Bethe ansatz

Now we will briefly review the algebraic Bethe ansatz (ABA from now on) for the inhomogeneous spin 1/2 Heisenberg spin chain. We will mostly follow notation and conventions in [1]. The core of the ABA is the quantum R-matrix, which is an operator that satisfies the Yang-Baxter equations. In the XXX Heisenberg chain it is given by

$$R(\lambda, \mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda, \mu) & c(\lambda, \mu) & 0 \\ 0 & c(\lambda, \mu) & b(\lambda, \mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad (2.10)$$

where $b(\lambda, \mu)$ and $c(\lambda, \mu)$ are functions of the rapidities λ and μ ,

$$b(\lambda, \mu) = \frac{\lambda - \mu}{\lambda - \mu + \eta} , \quad c(\lambda, \mu) = \frac{\eta}{\lambda - \mu + \eta} , \quad (2.11)$$

where η is the so called crossing parameter, which we will take to be i . The monodromy matrix of the spin chain is constructed as an ordered product of R-matrices,

$$T_0(\lambda) = R_{0L}(\lambda, \xi_L) \dots R_{01}(\lambda, \xi_1) , \quad (2.12)$$

and can be represented as a 2×2 matrix,

$$T_0(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = \prod_{n=L}^1 R_{0,n} . \quad (2.13)$$

The variables ξ_n are called the inhomogeneities of the spin chain. For the case of an homogeneous spin chain, which is the case that we are going to study during most of this paper, we have $\xi_n = \xi = \frac{i}{2}$ for any value of n . In order to solve the system we need to diagonalize the trace of the monodromy matrix, which is known as the transfer matrix, for any value of the rapidity λ . In the ABA this is done by first assuming the existence of a pseudo-vacuum $|0\rangle$. In the XXX Heisenberg chain this is just the completely ferromagnetic state, with all spins up, and can be represented by the tensor product

$$|0\rangle = \bigotimes_{n=1}^L |0\rangle_n, \quad \text{where} \quad |0\rangle_n = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_n. \quad (2.14)$$

The elements of the monodromy matrix act on the pseudo-vacuum as

$$A(\lambda)|0\rangle = a(\lambda)|0\rangle, \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle, \quad C(\lambda)|0\rangle = 0. \quad (2.15)$$

In this paper we will normalize the eigenvalues $a(\lambda)$ and $d(\lambda)$ as in reference [1],¹

$$a(\lambda) = 1, \quad d(\lambda) = \frac{(\lambda - \xi)^L}{(\lambda + \xi)^L}. \quad (2.16)$$

Some important relations that we are going to need along the calculations below are the commutation relations between the elements of the monodromy matrix [3],

$$\begin{aligned} (A + D)(\mu)B(\lambda) &= B(\lambda)(A + D)(\mu) + \frac{ic}{\lambda - \mu} [B(\lambda)(D - A)(\mu) - B(\mu)(D - A)(\lambda)], \\ C(\lambda)(A + D)(\mu) &= (A + D)(\mu)C(\lambda) + \frac{ic}{\lambda - \mu} [(D - A)(\mu)C(\lambda) - (D - A)(\lambda)C(\mu)], \\ [C(\lambda), B(\mu)] &= \frac{ic}{\lambda - \mu} (A(\lambda)D(\mu) - A(\mu)D(\lambda)). \end{aligned} \quad (2.17)$$

The parameter c depends on the model and can be fixed for instance by comparing these expressions with those in [24]. In the case of the Heisenberg chain we have to set $c = -1$. Note that these commutation relations diverge when λ and μ are equal. We will show later on how to regularize this apparent divergence.

A natural procedure to construct the eigenstates of the transfer matrix is to apply the operator $B(\lambda)$ on the pseudo-vacuum. Using the previous commutation relations one can

¹Note that with this choice $d(\xi) = 0$. In the general case of an inhomogeneous chain $d(\lambda)$ is given by

$$d(\lambda) = \prod_{n=1}^L \frac{(\lambda - \xi_n)}{(\lambda - \xi_n + i)},$$

and thus $d(\xi_n) = 0$ for any n .

find the action of the operator $(A + D)(\mu)$ on $B(\lambda)|0\rangle$ and impose that these states must be eigenstates of this operator,

$$(A + D)(\mu) \prod_{i=1}^N B(\lambda_i) |0\rangle = \tau(\mu, \{\lambda\}) \prod_{i=1}^N B(\lambda_i) |0\rangle \Rightarrow \frac{a(\lambda_i)}{d(\lambda_i)} \prod_{j \neq i} \frac{\lambda_i - \lambda_j - i}{\lambda_i - \lambda_j + i} = 1, \quad (2.18)$$

which are the Bethe ansatz equations for the ABA. If we want to recover the Bethe equations for the CBA in the previous subsection we need to write the momentum of the magnons as a function of the rapidity,²

$$\frac{\lambda_j - \xi}{\lambda_j + \xi} = e^{ip_j} \longleftrightarrow \lambda(p) = -\frac{1}{2} \cot\left(\frac{p}{2}\right). \quad (2.19)$$

If we perform this substitution, the S-matrix becomes

$$S_{ij} = \frac{\lambda_j - \lambda_i + i}{\lambda_j - \lambda_i - i}, \quad (2.20)$$

and equation (2.18) agrees with the Bethe equations in the CBA, (2.8).

In this paper we will also be interested in an extension of the above Bethe equations for the homogeneous Heisenberg spin chain. This extension was first introduced to reproduce the dispersion relation of the dilatation operator beyond one-loop in the $SU(2)$ sector of $\mathcal{N} = 4$ supersymmetric Yang-Mills. The resulting system contains long-range interactions, and requires deforming the rapidities to [23]

$$\lambda(p) = -\frac{1}{2} \cot\left(\frac{p}{2}\right) \sqrt{1 + 8g^2 \sin^2\left(\frac{p}{2}\right)}. \quad (2.21)$$

The long-range or asymptotic Bethe ansatz equations are given by

$$e^{ip_i L} \prod_{j \neq i}^N \frac{\lambda(p_i) - \lambda(p_j) + i}{\lambda(p_i) - \lambda(p_j) - i} = 1. \quad (2.22)$$

Inverting relation (2.21) we can present these equations in a more convenient way

$$\frac{x(\lambda_i + i/2)^L}{x(\lambda_i - i/2)^L} = \prod_{j \neq i}^N \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}, \quad (2.23)$$

where $x(\lambda)$ is given by

$$x(\lambda) = \frac{1}{2}\lambda + \frac{1}{2}\sqrt{\lambda^2 - 2g^2}. \quad (2.24)$$

²We follow the definition of the momentum in [1]. This definition can be related to the usual choice of momentum by a parity transformation that exchanges $p \rightarrow -p$ and $x \rightarrow -x$. As a consequence wherever we obtain a factor px it will be invariant under parity. Note that the Bethe ansatz is also invariant under this transformation.

In fact, the homogeneous long-range spin chain can be mapped to a short-range inhomogeneous spin chain by writing

$$\frac{P_L(\lambda_i + i/2)}{P_L(\lambda_i - i/2)} = \prod_{j \neq i}^N \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i} , \quad (2.25)$$

where the polynomial $P_L(\lambda)$ is given by

$$P_L(\lambda) = \prod_{n=1}^L (\lambda - \xi_n) , \quad \text{with} \quad \xi_n = \frac{i}{2} + \sqrt{2}g \cos \frac{(2n-1)\pi}{2L} , \quad (2.26)$$

that is, the system becomes a spin chain with inhomogeneities located at ξ_n .

2.3 Inverse scattering and inner products of Bethe states

Now we will briefly review the inverse scattering method and the scalar product of Bethe states which are the last two tools we are going to need in order to evaluate correlation functions using the ABA.

The inverse scattering problem

The first tool we will need is the solution to the inverse scattering problem, that is, the relations between the entries of the monodromy matrix and the local spin operators appearing in the CBA. This was found in [1] using the method of factorizing F-matrices for the case of the general XXZ inhomogeneous spin chain and later in [25] for the XYZ homogeneous spin chain using the properties of the R-matrix and the monodromy matrix. The solution for the inhomogeneous spin chain is

$$\sigma_k^+ = \prod_{i=1}^{k-1} (A + D)(\xi_i) C(\xi_k) \prod_{i=k+1}^L (A + D)(\xi_i) , \quad (2.27)$$

$$\sigma_k^- = \prod_{i=1}^{k-1} (A + D)(\xi_i) B(\xi_k) \prod_{i=k+1}^L (A + D)(\xi_i) , \quad (2.28)$$

$$\sigma_k^z = \prod_{i=1}^{k-1} (A + D)(\xi_i) (A - D)(\xi_k) \prod_{i=k+1}^L (A + D)(\xi_i) , \quad (2.29)$$

where k is a given site of the spin chain. As we will show, these expressions will allow to calculate expectation values of local operators by means of the Yang-Baxter algebra.

Scalar products

The second tool we will need is the value of scalar products of a Bethe state with an arbitrary state. There is a large amount of literature devoted to this kind of computation (see, for example, [3] and references therein). In [1] the scalar products were constructed by the action of the operators $B(\lambda)$ on the pseudo-vacuum

$$S_N(\{\mu_j\}, \{\lambda_k\}) = S_N(\{\lambda_k\}, \{\mu_j\}) = \langle 0 | \prod_{j=1}^N C(\mu_j) \prod_{k=1}^N B(\lambda_k) | 0 \rangle , \quad (2.30)$$

where the set of rapidities $\{\lambda_k\}$ is a solution to the Bethe equations and $\{\mu_j\}$ is an arbitrary set of parameters. The product can be represented as a ratio of two determinants,

$$S_N(\{\mu_j\}, \{\lambda_k\}) = \frac{\det T}{\det V} , \quad (2.31)$$

where T and V are $N \times N$ matrices given by

$$\begin{aligned} T_{ab} &= \frac{\partial \tau(\mu_b, \{\lambda\})}{\partial \lambda_a} , & \tau(\mu, \{\lambda\}) &= a(\mu) \prod_{k=1}^N \frac{\lambda_k - \mu + i}{\lambda_k - \mu} + d(\mu) \prod_{k=1}^N \frac{\lambda_k - \mu - i}{\lambda_k - \mu} , \\ V_{ab} &= \frac{1}{\mu_b - \lambda_a} , & \det V &= \frac{\prod_{a < b} (\lambda_a - \lambda_b) \prod_{j < k} (\mu_k - \mu_j)}{\prod_{k=1}^N \prod_{a=1}^N (\mu_k - \lambda_a)} . \end{aligned} \quad (2.32)$$

In an equivalent derivation we could have assumed that the set $\{\mu_j\}$ is a solution to the Bethe equations and that $\{\lambda_k\}$ is an arbitrary set of parameters.

If we take the limit $\mu_a \rightarrow \lambda_a$ in these expressions we recover the Gaudin formula for the square of the norm of a Bethe state [3],

$$\begin{aligned} S_N(\{\lambda_k\}, \{\lambda_k\}) &= i^N \prod_{j \neq k} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k} \det \Phi'(\{\lambda_k\}) , \\ \Phi'_{ab}(\{\lambda_k\}) &= -\frac{\partial}{\partial \lambda_b} \ln \left(\frac{a(\lambda_a)}{d(\lambda_a)} \prod_{b \neq a} \frac{\lambda_a - \lambda_b + i}{\lambda_a - \lambda_b - i} \right) . \end{aligned} \quad (2.33)$$

This way of calculating scalar products is valid for the case of a finite spin chain. The generalization of these expressions to the thermodynamical limit of very long chains can be found, for example, in reference [26].

2.4 Normalization of states

States in the algebraic and the coordinate Bethe ansatz are normalized differently. As a consequence, any correlation function evaluated using the ABA will differ from the

corresponding CBA computation by some global factor. The simplest correlation function that exhibits this issue is $\langle \lambda | \sigma_k^+ \sigma_l^- | \lambda \rangle$. In the CBA this correlation function is given by $e^{ip(l-k)}$. In order to approach the calculation of this correlator in the ABA we just need to write the spin operators in terms of elements of the monodromy matrix,

$$\begin{aligned} \langle \lambda | \sigma_k^+ \sigma_l^- | \lambda \rangle &= \langle 0 | C(\lambda) (A + D)^{k-1} B(\xi) (A + D)^{L-k+l-1} C(\xi) (A + D)^{L-l} B(\lambda) | 0 \rangle \\ &= e^{-ip(L-l+k-1)} \langle 0 | C(\lambda) B(\xi) (A + D)^{L-k+l-1} C(\xi) B(\lambda) | 0 \rangle . \end{aligned} \quad (2.34)$$

From the commutation relations (2.17) we find

$$\langle 0 | C(\lambda) B(\xi) = i \frac{d(\lambda)}{\lambda - \xi} \langle 0 | , \quad (2.35)$$

with an identical result for $C(\xi) B(\lambda) | 0 \rangle$. Recalling that the Bethe ansatz equation for the single-magnon state reads $d(\lambda) = 1$ we conclude that

$$\langle \lambda | \sigma_k^+ \sigma_l^- | \lambda \rangle = \frac{i^2 e^{ip(l-k+1)}}{(\lambda - \xi)^2} . \quad (2.36)$$

We can try to solve the disagreement with the CBA dividing this result by the norm of the state. This can be easily computed using the Gaudin formula (2.33),

$$\langle \lambda | \lambda \rangle = i \frac{\partial d}{\partial \lambda} = \frac{i^2 L}{\lambda^2 - \xi^2} . \quad (2.37)$$

Therefore

$$\frac{\langle \lambda | \sigma_k^+ \sigma_l^- | \lambda \rangle}{\langle \lambda | \lambda \rangle} = \frac{e^{ip(l-k)}}{L} \left(\frac{\lambda + \xi}{\lambda - \xi} e^{ip} \right) = \frac{e^{ip(l-k)}}{L} , \quad (2.38)$$

which is the result in the CBA provided we divide by the norm of the state in there. Thus we conclude that indeed there is a problem related to the normalization of Bethe states.

However in general the prescription of dividing the correlation function by the norm of the states is not enough to cure the disagreement. We can easily exhibit this if for instance we calculate the form factor $\langle 0 | \sigma_k^+ | \lambda \rangle$ and divide by $\sqrt{\langle \lambda | \lambda \rangle}$,

$$\frac{\langle 0 | \sigma_k^+ | \lambda \rangle}{\sqrt{\langle \lambda | \lambda \rangle}} = \frac{e^{ipk}}{\sqrt{L}} \sqrt{\frac{\lambda + \xi}{\lambda - \xi}} = \frac{e^{ip(k-\frac{1}{2})}}{\sqrt{L}} . \quad (2.39)$$

The reason for the additional $1/2$ factor is that besides the different normalization there is also an additional phase which depends on the rapidity. ³

³See reference [7] for a discussion on this point.

In order to fix the normalization of states in the ABA with respect to the normalization of states in the CBA we will go back to the definition of the transfer matrix, equation (2.13), and apply it to the ground state,

$$R_{0,n} |0\rangle_n = \begin{pmatrix} 1 & \frac{i}{\lambda-\xi+i} S_n^- \\ 0 & \frac{\lambda-\xi+i}{\lambda-\xi} \end{pmatrix} |0\rangle_n . \quad (2.40)$$

If we focus on the operator $B(\lambda)$, we can write

$$\begin{aligned} B(\lambda) &= \frac{i}{\lambda+\xi} \left[S_1^- + S_2^- \left(\frac{\lambda-\xi}{\lambda+\xi} \right) + S_3^- \left(\frac{\lambda-\xi}{\lambda+\xi} \right)^2 + \dots \right] \\ &= \frac{i}{\lambda+\xi} \sum_{n=1}^L S_n^- \left(\frac{\lambda-\xi}{\lambda+\xi} \right)^{n-1} = \frac{i}{\lambda-\xi} \sum_{n=1}^L S_n^- e^{ipn} . \end{aligned} \quad (2.41)$$

Therefore states with a single magnon in the ABA, $|\lambda\rangle^a$, relate to states in the CBA through

$$B(\lambda) |0\rangle = |\lambda\rangle^a = \frac{i}{\lambda-\xi} |\lambda\rangle^c . \quad (2.42)$$

When we repeat this with the state ${}^a\langle\lambda|$ we conclude that

$${}^a\langle\lambda| = i \frac{d(\lambda)}{\lambda+\xi} {}^c\langle\lambda| , \quad (2.43)$$

because for bra states

$${}_n\langle 0| R_{0,n} = {}_n\langle 0| \begin{pmatrix} \frac{\lambda-\xi+i}{\lambda-\xi} & 0 \\ \frac{i}{\lambda-\xi+i} S_n^+ & 1 \end{pmatrix} , \quad (2.44)$$

and therefore

$$\begin{aligned} C(\lambda) &= \frac{i}{\lambda+\xi} \left[S_1^+ \left(\frac{\lambda-\xi}{\lambda+\xi} \right)^{L-1} + S_2^+ \left(\frac{\lambda-\xi}{\lambda+\xi} \right)^{L-2} + S_3^+ \left(\frac{\lambda-\xi}{\lambda+\xi} \right)^{L-3} + \dots \right] \\ &= \frac{i d(\lambda)}{\lambda+\xi} \sum_{n=1}^L S_n^+ \left(\frac{\lambda-\xi}{\lambda+\xi} \right)^{-n} = \frac{i d(\lambda)}{\lambda+\xi} \sum_{n=1}^L S_n^+ e^{-ipn} . \end{aligned} \quad (2.45)$$

An identical discussion holds in the case of states with more than one magnon, so in general we conclude that

$$|\lambda_1, \lambda_2, \dots, \lambda_N\rangle^a = \prod_{j=1}^N \frac{i}{(\lambda_j - \xi)} \prod_{i < j} \frac{\lambda_j - \lambda_i + i}{\lambda_j - \lambda_i} |\lambda_1, \lambda_2, \dots, \lambda_N\rangle^c , \quad (2.46)$$

$$\langle\lambda_1, \lambda_2, \dots, \lambda_N|^a = \prod_{j=1}^N i \frac{d(\lambda_j)}{(\lambda_j + \xi)} \prod_{i < j} \frac{\lambda_j - \lambda_i - i}{\lambda_j - \lambda_i} \langle\lambda_1, \lambda_2, \dots, \lambda_N|^c . \quad (2.47)$$

The first factor can be removed by an appropriate normalization of the states, and thus there will only remain a shift in the position of the coordinates by $-\frac{1}{2}$. The second factor is related to the fact that CBA states are not symmetric if we interchange two magnons. In fact they pick up a phase which is equal to the S-matrix. On the other hand ABA states are symmetric under exchange of two magnons. Therefore if we want to obtain the same result from the CBA and the ABA we will have to normalize carefully the states. This can be done if we choose the phase in such a way that the correlation functions have the structure $\sqrt{\prod_{\mu_i < \mu_j} S_{ij}} \cdot \{\text{term symmetric in the rapidities}\}$, for reasons we will explain later. Despite being a very ad hoc solution, we are going to keep this idea in mind.

An alternative argument can be obtained if instead of using *B-states* to define the excitations we use *Z-states*, where

$$Z(\lambda) = B(\lambda)A^{-1}(\lambda) . \quad (2.48)$$

In fact it is natural to use these states because they generate a Zamolodchikov-Faddeev algebra [24],

$$Z(\lambda)Z(\mu) = Z(\mu)Z(\lambda)S_{\mu\lambda} = Z(\mu)Z(\lambda)\frac{\mu - \lambda - i}{\mu - \lambda + i} . \quad (2.49)$$

In this way states in the ABA will have the same behavior under the exchange of two magnons as states in the CBA.

In order to be able to work with *Z-states* we will have first to calculate the commutation relation between the operator A^{-1} and the *B* operator. To find this commutator we will start by taking the commutation relations between *A* and *B*,

$$\begin{aligned} A(\lambda)B(\mu) &= \left(1 - \frac{i}{\lambda - \mu}\right) B(\mu)A(\lambda) + \frac{i}{\lambda - \mu} B(\lambda)A(\mu) , \\ B(\mu)A(\lambda) &= \left(1 + \frac{i}{\lambda - \mu}\right) A(\lambda)B(\mu) - \frac{i}{\lambda - \mu} A(\mu)B(\lambda) . \end{aligned}$$

Now if we left and right-multiply both expressions by $A^{-1}(\lambda)$, and commute a factor $A(\mu)B(\lambda)$ arising in the second equation, we obtain

$$\begin{aligned} B(\mu)A^{-1}(\lambda) &= \frac{\lambda - \mu - i}{\lambda - \mu} A^{-1}(\lambda)B(\mu) + \frac{i}{\lambda - \mu} A^{-1}(\lambda)B(\lambda)A(\mu)A^{-1}(\lambda) , \\ A^{-1}(\lambda)B(\mu) &= \frac{\lambda - \mu}{\lambda - \mu - i} B(\mu)A^{-1}(\lambda) - \frac{i}{\lambda - \mu - i} A^{-1}(\lambda)B(\lambda)A(\mu)A^{-1}(\lambda) . \end{aligned}$$

We also need the action of A^{-1} over the vacuum state, which can be easily proven to be trivial. We thus conclude that there is a relationship between the *Z-states* and the

B -states,

$$A^{-1}(\lambda) \prod_i B(\mu_i) |0\rangle = \prod_i \frac{\lambda - \mu}{\lambda - \mu - i} \prod_i B(\mu_i) |0\rangle . \quad (2.50)$$

where we have used that if we have two magnons with the same rapidity the state must vanish. Therefore

$$\mathcal{R} \left[\prod_i Z(\mu_i) |0\rangle \right] = \prod_{i < j} \frac{\mu_j - \mu_i}{\mu_j - \mu_i + i} \prod_i B(\mu_i) |0\rangle , \quad (2.51)$$

where \mathcal{R} denotes just an ordering operator in the rapidities. Hence using the Zamolodchikov-Faddeev states instead of the usual magnon states introduces a phase shift. In fact this phase is the factor we wanted to introduce ad hoc.

However there could still be a problem if the norm of our states behaves in the same way. We can exclude this possibility if we introduce the operators $F(\lambda) = d(\lambda) D^{-1}(\lambda) C(\lambda)$. To prove that this is the operator we need in order to define the correct left-state, we first have to calculate the commutation relations of D with C . Using the same procedure as before we find that

$$\begin{aligned} D^{-1}(\lambda) C(\mu) &= \frac{\mu - \lambda - i}{\mu - \lambda} C(\mu) D^{-1}(\lambda) - \frac{i}{\lambda - \mu} D^{-1}(\lambda) D(\mu) C(\lambda) D^{-1}(\lambda) , \\ C(\mu) D^{-1}(\lambda) &= \frac{\mu - \lambda}{\mu - \lambda - i} D^{-1}(\lambda) C(\mu) - \frac{i}{\mu - \lambda - i} D^{-1}(\lambda) D(\mu) C(\lambda) D^{-1}(\lambda) . \end{aligned}$$

With these equations at hand we can easily prove that F generates a Zamolodchikov-Faddeev algebra, $F(\lambda)F(\mu) = F(\mu)F(\lambda)S_{\mu\lambda}$, and also that

$$\langle 0 | F(\mu) F(\lambda) = \frac{\mu - \lambda}{\mu - \lambda - i} \langle 0 | C(\lambda) C(\mu) , \quad (2.52)$$

so that

$$\langle 0 | F(\mu) F(\lambda) Z(\lambda) Z(\mu) | 0 \rangle = (\mu - \lambda)^2 \frac{\langle 0 | C(\mu) C(\lambda) B(\lambda) B(\mu) | 0 \rangle}{(\mu - \lambda - i)(\mu - \lambda + i)} , \quad (2.53)$$

which is symmetric under exchange of λ and μ as we wanted.

3 Form factors

We will now apply the tools introduced in the previous section to evaluate form factors of spin operators. In particular we will compute the three-magnon form factor $\langle \lambda | \sigma_k^+ | \mu_1 \mu_2 \rangle$. The extension to form factors with $n - 1$ outgoing magnons and n ingoing magnons is an

immediate extension of the computation below. Using relation (2.27) we can bring the problem to a computation in the ABA,

$$\begin{aligned}\langle \lambda | \sigma_k^+ | \mu_1 \mu_2 \rangle^a &= \langle 0 | C(\lambda) (A + D)^{k-1}(\xi) C(\xi) (A + D)^{L-k}(\xi) B(\mu_1) B(\mu_2) | 0 \rangle \\ &= e^{-i[(p_1+p_2) \cdot (L-k) + p_\lambda(k-1)]} \langle 0 | C(\lambda) C(\xi) B(\mu_1) B(\mu_2) | 0 \rangle .\end{aligned}\quad (3.1)$$

Note that although λ satisfies the Bethe equations for a single-magnon state, the pair $\{\lambda, \xi\}$ does not define a Bethe state. Therefore to find this form factor we need to calculate the scalar product of an arbitrary vector with a Bethe state. This can be done following the recipe we stated in section 2.3. The first step is to write (recall that $\xi = i/2$ for the Heisenberg chain)

$$\begin{aligned}\tau(\xi, \{\mu_1, \mu_2\}) &= \frac{\mu_1 - \xi + i}{\mu_1 - \xi} \frac{\mu_2 - \xi + i}{\mu_2 - \xi} = \frac{\mu_1 + \xi}{\mu_1 - \xi} \frac{\mu_2 + \xi}{\mu_2 - \xi} , \\ \tau(\lambda, \{\mu_1, \mu_2\}) &= \frac{\mu_1 - \lambda + 2\xi}{\mu_1 - \lambda} \frac{\mu_2 - \lambda + 2\xi}{\mu_2 - \lambda} + d(\lambda) \frac{\mu_1 - \lambda - 2\xi}{\mu_1 - \lambda} \frac{\mu_2 - \lambda - 2\xi}{\mu_2 - \lambda} ,\end{aligned}\quad (3.2)$$

so that the T and V matrices are given by

$$\begin{aligned}T_{11} &= \frac{-2\xi}{(\mu_1 - \xi)^2} \frac{\mu_2 + \xi}{\mu_2 - \xi} , \quad T_{21} = \frac{\mu_1 + \xi}{\mu_1 - \xi} \frac{-2\xi}{(\mu_2 - \xi)^2} , \\ T_{12} &= \frac{-2\xi}{(\mu_1 - \lambda)^2} \frac{\mu_2 - \lambda + 2\xi}{\mu_2 - \lambda} + \frac{2\xi}{(\mu_1 - \lambda)^2} \frac{\mu_2 - \lambda - 2\xi}{\mu_2 - \lambda} , \\ T_{22} &= \frac{\mu_1 - \lambda + 2\xi}{\mu_1 - \lambda} \frac{-2\xi}{(\mu_2 - \lambda)^2} + \frac{\mu_1 - \lambda - 2\xi}{\mu_1 - \lambda} \frac{2\xi}{(\mu_2 - \lambda)^2} , \\ \frac{1}{V} &= \frac{(\mu_1 - \xi)(\mu_1 - \lambda)(\mu_2 - \xi)(\mu_2 - \lambda)}{(\lambda - \xi)(\mu_1 - \mu_2)} ,\end{aligned}\quad (3.3)$$

where we have used that for a single-magnon the Bethe ansatz equations imply $d(\lambda) = 1$. After some immediate algebra the form factor becomes

$$\langle \lambda | \sigma_k^+ | \mu_1 \mu_2 \rangle^a = \frac{16\xi^3 e^{i(p_1+p_2-p_\lambda)k}}{(\lambda + \xi)(\mu_1 - \mu_2)} \left[\frac{\mu_2 + \xi}{(\mu_1 - \xi)(\mu_2 - \lambda)} - \frac{\mu_1 + \xi}{(\mu_2 - \xi)(\mu_1 - \lambda)} \right] .\quad (3.4)$$

Now if we want to read this result in the normalization of the CBA we need to recall the discussion in section 2.4. In the case at hand

$$\langle \lambda | \sigma_k^+ | \mu_1 \mu_2 \rangle^a = \frac{i d(\lambda)}{\lambda + \xi} \frac{\mu_2 - \mu_1 + i}{\mu_1 - \mu_2} \frac{1}{(\mu_1 - \xi)(\mu_2 - \xi)} \langle \lambda | \sigma_k^+ | \mu_1 \mu_2 \rangle^c .\quad (3.5)$$

Therefore

$$\langle \lambda | \sigma_k^+ | \mu_1 \mu_2 \rangle^c = e^{i(p_1+p_2-p_\lambda)k} \frac{-2}{\mu_2 - \mu_1 + i} \left[\frac{\mu_2^2 - \xi^2}{(\mu_2 - \lambda)} - \frac{\mu_1^2 - \xi^2}{(\mu_1 - \lambda)} \right] .\quad (3.6)$$

Now we have to divide by the norm of the states in both cases, which can be easily calculated using the Gaudin formula (2.33). In the ABA,

$$\langle \mu_1, \mu_2 | \mu_1, \mu_2 \rangle^a = \frac{16\xi^4 L^2 [(\mu_2 - \mu_1)^2 - 4\xi^2]}{(\mu_2 - \mu_1)^2 (\mu_1^2 - \xi^2) (\mu_2^2 - \xi^2)} \left(1 - \frac{2}{L} \cdot \frac{(\mu_1^2 + \mu_2^2 - 2\xi^2)}{[(\mu_2 - \mu_1)^2 - 4\xi^2]} \right). \quad (3.7)$$

Recalling again section 2.4, states in the CBA and the ABA are related through

$$\langle \mu_1, \mu_2 | \mu_1, \mu_2 \rangle^a = \left(\frac{\mu_2 - \mu_1 + i}{\mu_1 - \mu_2} \right) \left(\frac{\mu_2 - \mu_1 - i}{\mu_1 - \mu_2} \right) \frac{\langle \mu_1, \mu_2 | \mu_1, \mu_2 \rangle^c}{(\mu_1^2 - \xi^2) (\mu_2^2 - \xi^2)}, \quad (3.8)$$

and thus we conclude that

$$\langle \mu_1, \mu_2 | \mu_1, \mu_2 \rangle^c = 16\xi^4 L^2 \left(1 - \frac{2}{L} \cdot \frac{(\mu_1^2 + \mu_2^2 - 2\xi^2)}{[(\mu_2 - \mu_1)^2 - 4\xi^2]} \right). \quad (3.9)$$

Therefore at leading order the norm contributes with a factor \sqrt{L} for each magnon and it does not contain any momentum dependence. The properly normalized form factor will be

$$\begin{aligned} \frac{\langle \lambda | \sigma_k^+ | \mu_1 \mu_2 \rangle^c}{\sqrt{\langle \lambda | \lambda \rangle^c \langle \mu_1, \mu_2 | \mu_1, \mu_2 \rangle^c}} &= \frac{e^{i(p_1 + p_2 - p_\lambda)k}}{\sqrt{L^3}} \frac{2}{\mu_2 - \mu_1 + i} \left[\frac{\mu_2^2 - \xi^2}{(\mu_2 - \lambda)} - \frac{\mu_1^2 - \xi^2}{(\mu_1 - \lambda)} \right] \\ &\times \left(1 - \frac{2}{L} \cdot \frac{(\mu_1^2 + \mu_2^2 - 2\xi^2)}{[(\mu_2 - \mu_1)^2 - 4\xi^2]} \right)^{-1/2}. \end{aligned} \quad (3.10)$$

At this point there are two important points we should stress. The first one is that the form factor in the CBA agrees with the computation in the ABA when using Zamolodchikov-Faddeev states if we also perform the change $k \rightarrow k - \frac{1}{2}$ and we include a global minus sign. The second one is that our expression for $\langle \lambda | \sigma_k^+ | \mu_1 \mu_2 \rangle$ (regardless of whether it is the algebraic or the coordinate), conveniently normalized, is *valid to all orders in L* provided that we use an expression for the rapidities valid to all orders in L . We can thus write the rapidities in terms of the momenta, $\mu = -\frac{1}{2} \cot\left(\frac{p}{2}\right)$ and expand in the length of the chain. In the single-magnon state the momentum is quantized as

$$p_\lambda = \frac{2\pi n_\lambda}{L}. \quad (3.11)$$

In the two-magnon state the solution to the Bethe equations can be expanded as

$$p_1 = \frac{2\pi n_1}{L} + \frac{4\pi}{L^2} \frac{n_1 n_2}{n_1 - n_2} + \mathcal{O}(L^{-3}), \quad p_2 = \frac{2\pi n_2}{L} - \frac{4\pi}{L^2} \frac{n_1 n_2}{n_2 - n_1} + \mathcal{O}(L^{-3}). \quad (3.12)$$

We conclude that for the case of $k = 1$

$$\begin{aligned} \langle \lambda | \sigma_{k=1}^+ | \mu_1 \mu_2 \rangle^c &= \frac{1}{\sqrt{L^3}} \frac{2n_\lambda(n_1 + n_2 - n_\lambda)}{(n_\lambda - n_1)(n_\lambda - n_2)} \left[1 + \frac{1}{L} \frac{1}{(n_1 - n_2)^2} [(n_1^2 + n_2^2) \right. \\ &\left. + \frac{4n_1^2 n_2^2}{(n_\lambda - n_1)(n_\lambda - n_2)} + 2i\pi(n_1 - n_2)(n_1^2 - n_2^2 + n_1 n_2 - n_\lambda(n_1 - n_2))] + \dots \right]. \end{aligned} \quad (3.13)$$

The leading order term in this expression is the three-particle form factor obtained in [27] using the CBA with one particle of momentum p_λ and two external particles of momenta p_1 and p_2 . In order to obtain the subleading term we need to take into account the $\mathcal{O}(L^{-3})$ contributions to p_1 and p_2 .

We can get a more compact result, valid to all order in L , if we take into account the trace condition (2.9). Then in the two-magnon state we have $\mu_1 = -\mu_2$, and the Bethe equations can be solved analytically,⁴

$$\mu_1 = -\mu_2 = -\frac{1}{2} \cot \left(\frac{n\pi}{L-1} \right), \quad n \in \mathbb{Z}. \quad (3.14)$$

Substituting we obtain

$$\begin{aligned} \frac{\langle \lambda | \sigma_k^+ | \mu, -\mu \rangle^c}{\langle \lambda | \lambda \rangle^c \langle \mu, -\mu | \mu, -\mu \rangle^c} &= \frac{e^{-ip_\lambda k}}{L\sqrt{(L-1)}} \frac{2\mu(\mu + \xi)}{\mu^2 - \lambda^2} \\ &= e^{-2\pi i n_\lambda k/L} \frac{\cot \left(\frac{n\pi}{L-1} \right)}{L\sqrt{(L-1)}} \frac{2 \left[\cot \left(\frac{n\pi}{L-1} \right) - i \right]}{\cot^2 \left(\frac{n\pi}{L-1} \right) - \cot^2 \left(\frac{n_\lambda \pi}{L} \right)}, \end{aligned} \quad (3.15)$$

where n and n_λ are integer numbers.

4 Correlation functions

In the previous section we have described how the ABA can be employed to calculate form factors for spin operators. Apparently the fatal flaw of this method seems to be the possibility to perform computations involving two or more operators. This is because this kind of correlation functions will have the general form

$$\langle 0 | \dots C(\xi) (A + D)^n(\xi) \dots | 0 \rangle.$$

Therefore, according to the algebra (2.17), whenever we try to commute the $(A + D)$ operators with the C operator a divergence should appear. In this section we are going to describe how to deal with these divergences. We will first show how to proceed in the most simple case, that is, when we only have the operator C at the left of the $(A + D)^n$ factor. Later on we will extend the computation to more general correlation functions involving additional factors.

⁴We impose the trace condition on the two-magnon state rather than on the three-magnon state, because in this later case the correlation function becomes zero.

4.1 Evaluation of $\langle 0 | \sigma_k^+ \sigma_k^- | 0 \rangle$

We are going to start by evaluating the correlation function $\langle 0 | \sigma_k^+ \sigma_k^- | 0 \rangle$. Using the CBA we know this correlation function equals one. It will take some time to find the same result using the ABA, but the computation will serve to exhibit some general features of the method. The starting point in the ABA are the relations between local spin operators in the CBA and the elements of the monodromy matrix. If we recall that $(A + D)(\xi_i) | 0 \rangle = | 0 \rangle$ for the Heisenberg chain, we need to evaluate

$$\langle 0 | \sigma_k^+ \sigma_k^- | 0 \rangle = \langle 0 | C(\xi)(A + D)^{L-1}(\xi) B(\xi) | 0 \rangle . \quad (4.1)$$

In order to evaluate this correlation function we need to commute the operators $(A + D)$ with C or B using equation (2.17). However, although it seems that when trying to commute $(A + D)^n$ we should obtain a pole of order n because of the divergence of the commutation relations when the two rapidities are equal, the residue turns to be zero for all n and the expression is finite. In order to understand this cancellation some care will be needed. Let us first introduce some notation. We will define

$$\mathcal{F}_n^L(\alpha, \delta) = \langle 0 | C(\xi + \alpha) \mathcal{O}(\delta) | 0 \rangle ,$$

$$\mathcal{F}_{n+1}^L(\alpha, \delta) = \lim_{\beta \rightarrow \alpha} \langle 0 | C(\xi + \alpha)(A + D)(\xi + \beta) \mathcal{O}(\delta) | 0 \rangle = \lim_{\beta \rightarrow \alpha} f_{n+1}^L(\alpha, \beta, \delta) , \quad (4.2)$$

where $\mathcal{O}(\delta)$ denotes any operator. The reason for the subindex n is that in all the cases that we will consider $\mathcal{O}(\delta)$ will include a factor $(A + D)^n$. Now using (2.17) we can write

$$\begin{aligned} \mathcal{F}_{n+1}^L(\alpha, \delta) &= [1 + d(\xi + \alpha)] \mathcal{F}_n^L(\alpha, \delta) + \lim_{\beta \rightarrow \alpha} \frac{i}{\beta - \alpha} \{ [d(\xi + \beta) - 1] \mathcal{F}_n^L(\alpha, \delta) \\ &\quad - [d(\xi + \alpha) - 1] \mathcal{F}_n^L(\beta, \delta) \} , \end{aligned} \quad (4.3)$$

Now if we expand in a Taylor series we find that all terms of order $1/(\beta - \alpha)$ cancel themselves. Therefore we can safely take the limit $\beta \rightarrow \alpha$ to get

$$\mathcal{F}_{n+1}^L(\alpha, \delta) = \left[1 + d(\xi + \alpha) + i \frac{\partial d}{\partial \lambda} \Big|_{\xi + \alpha} \right] \mathcal{F}_n^L(\alpha, \delta) + i [1 - d(\xi + \alpha)] \frac{\partial \mathcal{F}_n^L(\alpha, \delta)}{\partial \alpha} . \quad (4.4)$$

We should stress that in this expression the derivative in α must be understood with respect to the argument of the C operator. As a consequence it does not act on the rest of the operators. This will introduce some subtleties in the next step of the calculation. The idea now is to use (4.4) as a recurrence equation to find $\langle 0 | \sigma_k^+ \sigma_k^- | 0 \rangle$. However this is not straightforward, because it requires information on correlation functions of the form

$$\langle 0 | C(\xi + \alpha)(A + D)(\xi + \delta) \dots | 0 \rangle , \quad (4.5)$$

but returns instead information about correlators of the form

$$\langle 0 | C(\xi + \alpha)(A + D)(\xi + \alpha)(A + D)(\xi + \delta) \dots | 0 \rangle . \quad (4.6)$$

Note that now also the argument of the first $(A + D)$ factor in (4.6) depends on α and thus in order to find the correlator we should take the derivative with respect to α in $f_{n+1}^L(\alpha, \beta, \delta)$, and then take the limit $\beta \rightarrow \alpha$, instead of taking directly the derivative in $\mathcal{F}_{n+1}^L(\alpha, \delta)$. Therefore using (4.3),

$$\begin{aligned} \lim_{\beta \rightarrow \alpha} \frac{\partial f_{n+1}^L(\alpha, \beta, \delta)}{\partial \alpha} &= [1 + d(\xi + \alpha)] \frac{\partial \mathcal{F}_n^L(\alpha, \delta)}{\partial \alpha} + \lim_{\beta \rightarrow \alpha} \frac{i}{\beta - \alpha} \left\{ [d(\xi + \beta) - 1] \frac{\partial \mathcal{F}_n^L(\alpha, \delta)}{\partial \alpha} \right. \\ &\quad - \frac{\partial d(\xi + \alpha)}{\partial \alpha} \mathcal{F}_n^L(\beta, \delta) + \frac{1}{(\beta - \alpha)^2} \left[[d(\xi + \beta) - 1] \mathcal{F}_n^L(\alpha, \delta) \right. \\ &\quad \left. \left. - [d(\xi + \alpha) - 1] \mathcal{F}_n^L(\beta, \delta) \right] \right\} . \end{aligned} \quad (4.7)$$

The remaining piece of the calculation is similar to the previous one. In this case after a series expansion we find a pole of order two and a pole of order one, but they cancel each other. The final result is

$$\begin{aligned} \lim_{\beta \rightarrow \alpha} \frac{\partial f_{n+1}^L(\alpha, \beta, \delta)}{\partial \alpha} &= [1 + d(\xi + \alpha)] \frac{\partial \mathcal{F}_n^L(\alpha, \delta)}{\partial \alpha} + \frac{i}{2} \frac{\partial^2 d}{\partial \alpha^2} \mathcal{F}_n^L(\alpha, \delta) \\ &\quad + \frac{i}{2} [1 - d(\xi + \alpha)] \frac{\partial^2 \mathcal{F}_n^L(\alpha, \delta)}{\partial \alpha^2} . \end{aligned} \quad (4.8)$$

So far we have proved that when we have one derivative and we commute one $(A + D)$ factor we get another derivative over the correlation function. In general if we have m derivatives we get

$$\begin{aligned} \lim_{\beta \rightarrow \alpha} \frac{\partial^m f_{n+1}^L(\alpha, \beta, \delta)}{\partial \alpha^m} &= [1 + d(\xi + \alpha)] \frac{\partial^m \mathcal{F}_n^L(\alpha, \delta)}{\partial \alpha^m} + \frac{i}{m+1} \frac{\partial^{m+1} d}{\partial \alpha^{m+1}} \mathcal{F}_n^L(\alpha, \delta) \\ &\quad + \frac{i}{m+1} [1 - d(\xi + \alpha)] \frac{\partial^{m+1} \mathcal{F}_n^L(\alpha, \delta)}{\partial \alpha^{m+1}} , \end{aligned} \quad (4.9)$$

that can be easily proved if we assume that the left-hand side of the equation has no poles. Under this assumption when we expand in a Taylor series we only need to track the terms without an $\beta - \alpha$,

$$\begin{aligned} \lim_{\beta \rightarrow \alpha} \frac{\partial^m f_{n+1}^L(\alpha, \beta, \delta)}{\partial \alpha^m} &= [1 + d(\xi + \alpha)] \frac{\partial^m \mathcal{F}_n^L(\alpha, \delta)}{\partial \alpha^m} \\ &\quad + \lim_{\beta \rightarrow \alpha} \frac{\partial^m}{\partial \alpha^m} \left\{ \frac{i}{\beta - \alpha} \left[(d(\xi + \beta) - 1) \mathcal{F}_n^L(\alpha, \delta) - (d(\xi + \alpha) - 1) \mathcal{F}_n^L(\beta, \delta) \right] \right\} . \end{aligned} \quad (4.10)$$

The second term on the right hand side of this expression can be written as

$$\lim_{\beta \rightarrow \alpha} \sum_j \binom{m}{j} \frac{i}{(\beta - \alpha)^{j+1} (j+1)} \cdot \left[\frac{\partial^{j+1} d}{\partial \alpha^{j+1}} \frac{\partial^{m-j} \mathcal{F}_n^L}{\partial \alpha^{m-j}} - \frac{\partial^{m-j} (d-1)}{\partial \alpha^{m-j}} \frac{\partial^{j+1} \mathcal{F}_n^L}{\partial \alpha^{j+1}} \right] (\beta - \alpha)^{j+1} + \dots ,$$

where the dots stand for terms proportional to $(\beta - \alpha)^k$. Now it is clear that the terms in j are canceled by the terms in $m - j - 1$. Therefore the only term surviving is the one with $j = m$, which does not have a partner. This is expression (4.9).

Let us summarize our results up to this point. We have obtained a complete set of recurrence equations

$$\begin{aligned} \mathcal{F}_{n+1}^L(\alpha) &= \left[1 + d(\xi + \alpha) + i \left. \frac{\partial d}{\partial \lambda} \right|_{\xi + \alpha} \right] \mathcal{F}_n^L(\alpha) + i [1 - d(\xi + \alpha)] \mathcal{D} \mathcal{F}_n^L(\alpha) , \\ \mathcal{D}^m \mathcal{F}_{n+1}^L(\alpha) &= [1 + d(\xi + \alpha)] \mathcal{D}^m \mathcal{F}_n^L(\alpha) + \frac{i}{m+1} \frac{\partial^{m+1} d}{\partial \alpha^{m+1}} \mathcal{F}_n^L(\alpha) \\ &\quad + \frac{i}{m+1} [1 - d(\xi + \alpha)] \mathcal{D}^{m+1} \mathcal{F}_n^L(\alpha) , \\ \mathcal{D}^m \mathcal{F}_0^L(\alpha) &= \frac{\partial^m \mathcal{F}_0^L(\alpha)}{\partial \alpha^m} , \quad \text{with} \quad \mathcal{F}(\alpha) = \lim_{\delta \rightarrow \alpha} \mathcal{F}(\alpha, \delta) , \end{aligned} \tag{4.11}$$

and where \mathcal{D} is just a convenient notation to refer both to the derivative and the limit,

$$\mathcal{D}^m \mathcal{F}(\alpha) = \lim_{\substack{\delta \rightarrow \alpha \\ \beta \rightarrow \alpha}} \frac{\partial^m f(\alpha, \beta, \delta)}{\partial \alpha^m} . \tag{4.12}$$

Now we are ready to calculate the correlation function provided a starting condition is given. In our case, using the last equation of (2.17),

$$\mathcal{F}_0^L(\alpha) = \langle 0 | C(\xi) B(\xi + \alpha) | 0 \rangle = -\frac{ic}{\alpha} \frac{\alpha^L}{(\alpha + i)^L} , \tag{4.13}$$

which takes values $\mathcal{F}_0^1(0) = -c = 1$ and $\mathcal{F}_0^{L>1} = 0$. In order to find $\langle 0 | \sigma_k^+ \sigma_k^- | 0 \rangle$ we have to calculate $\mathcal{F}_{L-1}^L(0)$. Because $\mathcal{F}_0^L(0)$ has a zero of order $L - 1$, the only terms that can contribute are those which involve a number of derivatives of $\mathcal{F}_0^L(\alpha)$ greater than or equal to $L - 1$ (other possible terms will require many more derivatives). In appendix A we will construct the correlation function $\mathcal{F}_n^L(\alpha)$ in full generality, but in this case it is easy to see that

$$\mathcal{F}_{L-1}^L(\alpha) = \frac{i^{L-1}}{(L-1)!} \frac{\partial^{L-1} \mathcal{F}_0^L(\alpha)}{\partial \alpha^{L-1}} + \dots = \frac{i^{L-1}}{(L-1)!} \cdot i \frac{(L-1)!}{i^L} + \mathcal{O}(\alpha) . \tag{4.14}$$

In the limit $\alpha \rightarrow 0$ we conclude that the value of this correlator is one, as expected from the CBA. We can also prove that $\mathcal{F}_n^L(\alpha) = 0$ for $0 \leq n < L - 1$, which also agrees with the result $\langle 0 | \sigma_k^+ \sigma_l^- | 0 \rangle = 0$ when $k \neq l$ of the CBA.

4.2 Evaluation of $\langle 0 | \sigma_k^+ \sigma_l^+ | \mu_1 \mu_2 \rangle$

We will now evaluate the correlation function $\langle 0 | \sigma_k^+ \sigma_l^+ | \mu_1 \mu_2 \rangle$. Using relation (2.27) we can bring again the problem to the ABA,

$$\langle 0 | \sigma_k^+ \sigma_l^+ | \mu_1 \mu_2 \rangle = \langle 0 | (A + D)^{k-1}(\xi) C(\xi) (A + D)^n(\xi) C(\xi) (A + D)^{L-l}(\xi) | \mu_1 \mu_2 \rangle , \quad (4.15)$$

where $n = L + l - k - 1$. The first factor $(A + D)$ acts trivially on the vacuum. On the contrary, the last factor $(A + D)$ acts on the two magnon state $|\mu_1 \mu_2\rangle = B(\mu_1) B(\mu_2) |0\rangle$ and provides a factor $e^{-i(p_1+p_2)\cdot(L-l)} = e^{i(p_1+p_2)l}$, where in the last equality we have used the periodicity condition for the Bethe roots. The contribution from the remaining factors can be obtained in a similar way to the previous correlation function. To continue with the notation introduced in that case, now we will name correlation functions with n inner factors of $(A + D)$ by $\mathcal{G}_n^L(\alpha)$,

$$\mathcal{G}_n^L(\alpha) = \langle 0 | C(\xi + \alpha) \mathcal{O}(\delta) C(\xi) B(\mu_1) B(\mu_2) | 0 \rangle . \quad (4.16)$$

As we will show, the problem can again be solved as a recurrence and thus the starting point will be to find the reference correlator

$$\mathcal{G}_0^L(\alpha) = \langle 0 | C(\xi + \alpha) C(\xi) B(\mu_1) B(\mu_2) | 0 \rangle = \langle 0 | \sigma_1^+ \sigma_L^+ | \mu_1 \mu_2 \rangle , \quad (4.17)$$

which is the product of a on-shell Bethe state with an off-shell Bethe state. As described in section 2 we can write

$$\langle 0 | C(\xi + \alpha) C(\xi) B(\mu_1) B(\mu_2) | 0 \rangle = \frac{\det T}{V} . \quad (4.18)$$

Now the functions $\tau(\xi)$ and $\tau(\xi + \alpha)$ are

$$\begin{aligned} \tau(\xi, \{\mu_1, \mu_2\}) &= \frac{\mu_1 - \xi + i}{\mu_1 - \xi} \frac{\mu_2 - \xi + i}{\mu_2 - \xi} = \frac{\mu_1 + \xi}{\mu_1 - \xi} \frac{\mu_2 + \xi}{\mu_2 - \xi} , \\ \tau(\xi + \alpha, \{\mu_1, \mu_2\}) &= \frac{\mu_1 + \xi - \alpha}{\mu_1 - \xi - \alpha} \frac{\mu_2 + \xi - \alpha}{\mu_2 - \xi - \alpha} + \frac{\alpha^L}{(i + \alpha)^L} \frac{\mu_1 - 3\xi - \alpha}{\mu_1 - \xi - \alpha} \frac{\mu_2 - 3\xi - \alpha}{\mu_2 - \xi - \alpha} , \end{aligned} \quad (4.19)$$

and thus the matrices T and V become

$$\begin{aligned} T_{11} &= \frac{-2\xi}{(\mu_1 - \xi)^2} \frac{\mu_2 + \xi}{\mu_2 - \xi} , \quad T_{21} = \frac{\partial \tau(\xi, \{\mu_1, \mu_2\})}{\partial \mu_2} = \frac{\mu_1 + \xi}{\mu_1 - \xi} \frac{-2\xi}{(\mu_2 - \xi)^2} , \\ T_{12} &= \frac{-2\xi}{(\mu_1 - \xi - \alpha)^2} \frac{\mu_2 + \xi - \alpha}{\mu_2 - \xi - \alpha} + \frac{\alpha^L}{(i + \alpha)^L} \frac{2\xi}{(\mu_1 - \xi - \alpha)^2} \frac{\mu_2 - 3\xi - \alpha}{\mu_2 - \xi - \alpha} , \\ T_{22} &= \frac{\mu_1 + \xi - \alpha}{\mu_1 - \xi - \alpha} \frac{-2\xi}{(\mu_2 - \xi - \alpha)^2} + \frac{\alpha^L}{(i + \alpha)^L} \frac{\mu_1 - 3\xi - \alpha}{\mu_1 - \xi - \alpha} \frac{2\xi}{(\mu_2 - \xi - \alpha)^2} , \\ \frac{1}{V} &= \frac{(\mu_1 - \xi)(\mu_1 - \xi - \alpha)(\mu_2 - \xi)(\mu_2 - \xi - \alpha)}{\alpha(\mu_1 - \mu_2)} . \end{aligned} \quad (4.20)$$

After some algebra we can easily organize $\mathcal{G}_0^L(\alpha)$ as an expansion in α ,

$$\begin{aligned}\mathcal{G}_0^L(\alpha) &= (A_0 + \alpha A_1 + \alpha^2 A_2 + \dots) + \alpha^{L-1} (B_{L-1} + \alpha B_L + \alpha^2 B_{L+1} + \dots) \\ &+ \alpha^{2L-1} (C_{2L-1} + \alpha C_{2L} + \alpha^2 C_{2L+1} + \dots) ,\end{aligned}\quad (4.21)$$

with A_q and B_{L+q-1} given by ⁵

$$\begin{aligned}A_q &= \frac{1}{\mu_1 - \mu_2} \frac{\mu_1^+ \mu_2^+}{\mu_1^- \mu_2^-} \left[\frac{1}{(\mu_1^-)^q} \frac{(\mu_2 - \mu_1 + i)}{\mu_1^- \mu_2^+} + \frac{1}{(\mu_2^-)^q} \frac{(\mu_2 - \mu_1 - i)}{\mu_1^+ \mu_2^-} \right] , \\ B_{L+q-1} &= \sum_{j=0}^q i^j \binom{L+j-1}{j} \beta_{q-j} ,\end{aligned}\quad (4.22)$$

where we have defined

$$\begin{aligned}\beta_0 &= B_{L-1} = \frac{1}{i^L} \frac{1}{\mu_1^- \mu_2^-} \frac{1}{\mu_1 - \mu_2} (\mu_2^+ \mu_1^{---} - \mu_1^+ \mu_2^{---}) , \\ \beta_q &= \frac{1}{i^L} \frac{1}{\mu_1 - \mu_2} \frac{1}{\mu_1^- \mu_2^-} \left(\frac{\mu_2^+ \mu_1^{---} - \mu_2^+ \mu_2^-}{(\mu_2^-)^q} - \frac{\mu_1^+ \mu_2^{---} - \mu_1^+ \mu_1^-}{(\mu_1^-)^q} \right) ,\end{aligned}\quad (4.23)$$

with $\mu_i^j = \mu_i + j\xi$ and $B_q = C_p = 0$ for $q < L-1$ and $p < 2L-1$. The next step is to find the general form of the correlation function $\mathcal{G}_n^L(\alpha)$. Using the recurrence equations (4.11) the first terms can be easily calculated for a general value of α ,

$$\begin{aligned}\mathcal{G}_1^L(\alpha) &= [1 + d + i \frac{\partial d}{\partial \lambda}] \mathcal{G}_0^L(\alpha) + i [1 - d] \frac{\partial \mathcal{G}_0^L(\alpha)}{\partial \lambda} , \\ \mathcal{G}_2^L(\alpha) &= [1 + 2d + 2i \frac{\partial d}{\partial \lambda} + 2id \frac{\partial d}{\partial \lambda} + d^2 - \left(\frac{\partial d}{\partial \lambda} \right)^2 - \frac{1}{2} \frac{\partial^2 d}{\partial \lambda^2} + \frac{d}{2} \frac{\partial^2 d}{\partial \lambda^2}] \mathcal{G}_0^L(\alpha) \\ &+ [2i - 2id^2 - \frac{\partial d}{\partial \lambda} + d \frac{\partial d}{\partial \lambda}] \frac{\partial \mathcal{G}_0^L(\alpha)}{\partial \lambda} - \frac{(1-d)^2}{2} \frac{\partial^2 \mathcal{G}_0^L(\alpha)}{\partial \lambda^2} ,\end{aligned}\quad (4.24)$$

where $d = d(\xi + \alpha)$ and $\frac{\partial d}{\partial \lambda} = \frac{\partial d}{\partial \lambda} \Big|_{\xi+\alpha}$. If we take now the limit $\alpha \rightarrow 0$, all the d and derivatives of d disappear, unless it is a derivative of d of order greater or equal to L . The computation of $\mathcal{G}_n^L(0)$ with arbitrary n is a little bit more involved. We have collected all details in appendix A. We find

$$\mathcal{G}_n^L(0) = \sum_{q=0}^n \binom{n}{q} \frac{i^q \mathcal{D}^q}{q!} \mathcal{G}_0^L(\alpha) \Big|_{\alpha=0} + \theta(n-L) \mathcal{G}_{n-L}^L(0) ,\quad (4.25)$$

where $\theta(x)$ is the Heaviside step function. If we use now expansion (4.21) and perform the derivatives we can write

$$\mathcal{G}_n^L(0) = \sum_{q=0}^n \binom{n}{q} i^q (A_q + B_q + C_q) + \theta(n-L) \mathcal{G}_{n-L}^L(0) .\quad (4.26)$$

Now we are finally ready to evaluate $\langle 0 | \sigma_k^+ \sigma_l^+ | \mu_1 \mu_2 \rangle$ for different values of n .

⁵Because of periodicity it is unnecessary to write the explicit expression for C .

The case $n < L - 1$

We will first consider the case where $n < L - 1$, which corresponds to $l < k$. From (4.26) it is clear that when $n < L - 1$ the only contribution is from the A_q terms, that can be easily summed up,

$$\sum_{q=0}^n \binom{n}{q} i^q A_q = \frac{1}{\mu_1 - \mu_2} \frac{\mu_1^+ \mu_2^+}{\mu_1^- \mu_2^-} \left[\left(\frac{\mu_1^+}{\mu_1^-} \right)^n \frac{(\mu_2 - \mu_1 + i)}{\mu_1^- \mu_2^+} + \left(\frac{\mu_2^+}{\mu_2^-} \right)^n \frac{(\mu_2 - \mu_1 - i)}{\mu_1^+ \mu_2^-} \right]. \quad (4.27)$$

Recalling now that the rapidities parametrize the momenta, $\mu_i^+/\mu_i^- = e^{-ip_i}$, equation (4.15) can be written as

$$\langle 0 | \sigma_k^+ \sigma_l^+ | \mu_1 \mu_2 \rangle = \frac{1}{\mu_1 - \mu_2} \frac{\mu_2 - \mu_1 + i}{\mu_1^- \mu_2^-} [e^{ip_1(k-L)+ip_2l} + e^{ip_2(k-L)+ip_1l} S_{21}] , \quad (4.28)$$

where we have inserted the S-matrix,

$$S_{21} = \frac{\mu_2 - \mu_1 - i}{\mu_2 - \mu_1 + i} , \quad (4.29)$$

and we have taken into account that $n = L + l - k - 1$. Using now the Bethe equations $e^{-ip_1L} = e^{ip_2L} = S$, we find

$$\langle 0 | \sigma_k^+ \sigma_l^+ | \mu_1 \mu_2 \rangle = \frac{1}{\mu_1 - \mu_2} \frac{\mu_2 - \mu_1 + i}{\mu_1^- \mu_2^-} [e^{i(p_1k+p_2l)} S_{21} + e^{i(p_2k+p_1l)}] . \quad (4.30)$$

Note that although this result is only true as long as $l < k$, we already find that it corresponds to what we should have obtained from the CBA up to the factor in front of the bracket. At the end of this section we will see how the normalization proposed in section 2.4 allows to get rid of it.

The case $n = L - 1$

Our next step is the calculation of $\mathcal{G}_{L-1}^L(0)$, which must be identically zero, because it corresponds to the case where both operators are located at the same site, $k = l$. If we take the equation (4.26), we find that this correlation function can be written as

$$\mathcal{G}_{L-1}^L(0) = i^{L-1} B_{L-1} + \sum_{q=0}^{L-1} \binom{n}{q} i^q A_q . \quad (4.31)$$

The second term is already known from the previous calculation. Therefore we only have to substitute the special value we are interested in and make use of the Bethe equations to get

$$\sum_{q=0}^{L-1} \binom{L-1}{q} i^q A_q = -\frac{2}{\mu_1^- \mu_2^-} . \quad (4.32)$$

On the other hand

$$i^{L-1}B_{L-1} = \frac{i}{\mu_1 - \mu_2} \frac{1}{\mu_1^- \mu_2^-} (\mu_1^+ \mu_2^{---} - \mu_1^{---} \mu_2^+) = \frac{2}{\mu_1^- \mu_2^-} . \quad (4.33)$$

Therefore $\mathcal{G}_{L-1}^L(0) = 0$, as we expected from the CBA.

The case $n > L - 1$

The last correlation functions that we will evaluate will be those with $L - 1 < n < 2L - 1$. Obviously, because of periodicity of the spin chain, we expect that $\mathcal{G}_{n+L}^L(0)$ should equal $\mathcal{G}_n^L(0)$. In order to prove this we will first show that the contribution from the B terms is going to be $\binom{n-L}{q+1} i^{L+q} \beta_q$. Next we will demonstrate that this coefficient cancels $\sum_{q=0}^n \binom{n}{q} i^q A_q$, and thus we will conclude that $\mathcal{G}_{n+L}^L(0) = \mathcal{G}_n^L(0)$. Let us see how it goes. Recalling the expression for B_q in (4.22) and performing the sum we find

$$\sum_{q=L-1}^n \binom{n}{q} i^q B_q = \sum_{s=0}^{n-L+1} \sum_{t=0}^s \binom{n}{s+L-1} \binom{L+t-1}{t} i^{s+t+L-1} \beta_{s-t} . \quad (4.34)$$

In order to obtain the coefficient of a particular β_q we have to set $s - t = q$ in the previous expression. For instance, the coefficient of β_q is

$$i^{L+q-1} \sum_{r=0}^{n-L-q+1} \binom{n}{L+r+q-1} \binom{L+r-1}{r} (-1)^r , \quad (4.35)$$

where we have taken $r = s - q$ because all terms with $s < q$ do not contribute to β_q . We can rewrite the sum and the binomial coefficients in a way that will allow us to use the definition of the hypergeometric function

$$\begin{aligned} & \frac{n!}{(L-1)!} \sum_{r=0}^{n-L-q+1} \frac{(L+r-1)!}{(L+r+q-1)!} \binom{n-L-q+1}{r} (-1)^r \\ &= {}_2F_1(L, q-1+L-n; L+q; 1) n! = \frac{(n-L)!}{(q-1)!} , \end{aligned} \quad (4.36)$$

where in the last equality we have used Kummer's first formula,

$${}_2F_1\left(\frac{1}{2} + m - q, -n; 2m + 1; 1\right) = \frac{\Gamma(2m+1)\Gamma(m+\frac{1}{2}+q+n)}{\Gamma(m+\frac{1}{2}+q)\Gamma(2m+1+n)} . \quad (4.37)$$

Therefore there is no contribution from β_0 . But the rest of the coefficients will contribute with $\binom{n-L}{q-1} i^{L+q-1}$. Now if we use that

$$\sum_{\alpha} \binom{K-L}{\alpha} \frac{i^{\alpha}}{(\mu^-)^{\alpha}} = \left(\frac{\mu^+}{\mu^-}\right)^{K-L} , \quad (4.38)$$

together with $\mu_1^+ \mu_2^{---} - \mu_1^+ \mu_1^- = \mu_1^+ (\mu_2 - \mu_1 - i)$, we find that

$$\begin{aligned} & \sum_{q=0}^{n-L} \binom{n-L}{q} i^{L+q} \beta_q \\ &= \frac{1}{\mu_1^- \mu_2^-} \frac{-1}{\mu_1 - \mu_2} \left[\left(\frac{\mu_1^+}{\mu_1^-} \right)^{n-L+1} (\mu_2 - \mu_1 - i) + \left(\frac{\mu_2^+}{\mu_2^-} \right)^{n-L+1} (\mu_2 - \mu_1 + i) \right]. \end{aligned} \quad (4.39)$$

If we now remove the $-L$ factor by extracting a factor S , expression (4.39) cancels exactly the contribution from the sum of the A 's in (4.27). Finally we conclude that

$$\begin{aligned} \langle 0 | \sigma_k^+ \sigma_l^+ | \mu_1 \mu_2 \rangle &= \frac{e^{i(p_1+p_2)l}}{\mu_1 - \mu_2} \frac{1}{\mu_1^- \mu_2^-} \left[e^{ip_1(k-l)} (\mu_2 - \mu_1 + i) + e^{ip_2(k-l)} (\mu_2 - \mu_1 - i) \right] \\ &= \frac{1}{\mu_1 - \mu_2} \frac{\mu_2 - \mu_1 + i}{\mu_1^- \mu_2^-} \left[e^{i(p_1 k + p_2 l)} + e^{i(p_2 k + p_1 l)} S_{21} \right], \end{aligned} \quad (4.40)$$

which agrees with (4.30), but with k and l exchanged because now we are in the case $k < l$.

We will end this section by normalizing properly the above correlation functions. Following the discussion in section 2.4,

$$\langle 0 | \sigma_k^+ \sigma_l^+ | \mu_1 \mu_2 \rangle^{\text{ZF}} = \frac{-1}{\mu_1^- \mu_2^-} \left[e^{i(p_1 k + p_2 l)} + e^{i(p_2 k + p_1 l)} S_{21} \right], \quad (4.41)$$

On the other hand, the norm of the states in the ABA is given by (3.7), while

$$\langle \mu_1, \mu_2 | \mu_1, \mu_2 \rangle^{\text{ZF}} = \frac{16\xi^4 L^2}{(\mu_1^2 - \xi^2)(\mu_2^2 - \xi^2)} \left(1 - \frac{2}{L} \cdot \frac{(\mu_1^2 + \mu_2^2 - 2\xi^2)}{[(\mu_2 - \mu_1)^2 - 4\xi^2]} \right). \quad (4.42)$$

Therefore, we conclude that

$$\begin{aligned} \frac{\langle 0 | \sigma_k^+ \sigma_l^+ | \mu_1 \mu_2 \rangle}{\sqrt{\langle \mu_1, \mu_2 | \mu_1, \mu_2 \rangle}} &= \frac{e^{ip_1(k-\frac{1}{2})+ip_2(l-\frac{1}{2})} + e^{ip_2(k-\frac{1}{2})+ip_1(l-\frac{1}{2})} S}{L} \\ &\times \left(1 - \frac{2}{L} \cdot \frac{(\mu_1^2 + \mu_2^2 - 2\xi^2)}{[(\mu_2 - \mu_1)^2 - 4\xi^2]} \right)^{-1/2}. \end{aligned} \quad (4.43)$$

Now, as in the case of the form factor calculated in the previous section, we can take into account the trace condition (2.9). When we replace the rapidities from equation (3.14) in these expressions, after some immediate algebra we obtain

$$L \frac{\langle 0 | \sigma_k^+ \sigma_l^+ | \mu, -\mu \rangle}{\sqrt{\langle \mu, -\mu | \mu, -\mu \rangle}} = 2 \sqrt{\frac{L}{L-1}} \cos \left(\frac{(2|l-k|-1)\pi n}{L-1} \right), \quad (4.44)$$

with $|l-k| \leq L-1$. This result extends the analysis in reference [6], where this correlation function was calculated for the cases $l-k=1$ and $l-k=2$ (we have written the factor L on the left hand side of (4.44) to follow conventions in there).

5 The long-range Bethe ansatz

In this section we are going to apply the method that we have developed along this paper to the long-range BDS spin chain [23]. This can be done quite easily because in all our previous expressions we have kept general the homogeneous point. Therefore, as the BDS spin chain can be mapped into an inhomogeneous short-range spin chain, with the inhomogeneities located at

$$\xi_n = \frac{i}{2} + \sqrt{2}g \cos \frac{(2n-1)\pi}{2L} \equiv \xi + g\kappa_n, \quad (5.1)$$

it is rather simple to extend all computations above to the long-range Bethe ansatz. An immediate example is the the normalization factor for the operator $B(\lambda)$, which is straightforward to calculate given the expressions from section 2.4,

$$B(\lambda) = \sum_{n=1}^L \frac{iS_n^-}{\lambda - \xi_n} \left(\prod_{l=1}^n \frac{\lambda - \xi_l}{\lambda - \xi_l + i} \right). \quad (5.2)$$

We conclude therefore that in the long-range Bethe ansatz the difference in normalization between the ABA and the CBA depends on the site where the spin operator acts. An analogous result follows for the operator $C(\lambda)$.

Another example of computation that we can readily extend to the long-range Bethe ansatz is the calculation of scalar products. This is immediate because the solution to the inverse scattering problem in expressions (2.27)-(2.29) is valid for an inhomogeneous spin chain. Furthermore equations (2.31) and (2.32) can be directly used without modifications. An immediate example is the calculation of the form factor of the single-magnon state,

$$\langle 0 | \sigma_k^+ | \lambda \rangle = \frac{i}{\lambda - \xi - g\kappa_k} \prod_{j=1}^k \frac{\lambda - \xi - g\kappa_j}{\lambda + \xi - g\kappa_j}, \quad (5.3)$$

which as in the case of the homogeneous spin chain should also be divided by the norm

$$\sqrt{\langle \lambda | \lambda \rangle} = \sqrt{i \frac{\partial d}{\partial \lambda}} = i \sqrt{\sum_{m=1}^L \frac{1}{(\lambda - \xi - g\kappa_m)(\lambda + \xi - g\kappa_m)}}. \quad (5.4)$$

The limit $g \rightarrow 0$ reduces to the result in section 2.4. In an identical way we can extend the analysis to the correlation functions obtained in section 4. For instance,

$$\begin{aligned} \langle 0 | \sigma_k^+ \sigma_{k+1}^+ | \mu_1 \mu_2 \rangle_Z &= \left[\frac{\mu_1 + \xi - g\kappa_k}{\mu_1 - \xi - g\kappa_{k+1}} \frac{\mu_2 + \xi - g\kappa_{k+1}}{\mu_2 - \xi - g\kappa_k} - (\mu_2 \leftrightarrow \mu_1) \right] \\ &\times \frac{1}{[g(\kappa_{k+1} - \kappa_k)(\mu_1 - \mu_2 - i)]} \prod_{j=1}^{k+1} \frac{\mu_1 - \xi - g\kappa_j}{\mu_1 + \xi - g\kappa_j} \frac{\mu_2 - \xi - g\kappa_j}{\mu_2 + \xi - g\kappa_j}. \end{aligned} \quad (5.5)$$

The norm is now given by

$$\sqrt{\langle \mu_1 \mu_2 | \mu_1 \mu_2 \rangle_Z} = \frac{2}{(\mu_2 - \mu_1)^2 - 4\xi^2} \sum_j \left[\frac{1}{(\mu_1 - g\kappa_j)^2 - 4\xi^2} + \frac{1}{(\mu_2 - g\kappa_j)^2 - 4\xi^2} \right] - \sum_j \sum_k \frac{1}{[(\mu_1 - g\kappa_j)^2 - 4\xi^2][(\mu_2 - g\kappa_k)^2 - 4\xi^2]} . \quad (5.6)$$

We should stress that an important difference when comparing with the homogeneous XXX Heisenberg spin chain in the previous sections is that because all the inhomogeneities are different the commutation of factors $(A + D)$ does not lead now to any divergences. Therefore we do not have to make use of the procedure we have developed along this paper. For instance, the correlation function $\langle 0 | \sigma_k^+ \sigma_{k+2}^+ | \mu_1 \mu_2 \rangle$ can be calculated by direct use of the commutation relations (2.17),

$$\begin{aligned} \langle 0 | \sigma_k^+ \sigma_{k+2}^+ | \mu_1 \mu_2 \rangle &= \langle 0 | C(\xi_k)(A + D)(\xi_{k+1})C(\xi_{k+2})B(\mu_1)B(\mu_2) | 0 \rangle p(k + 2) \\ &= \left[\frac{\xi_k - \xi_{k+1} + i}{\xi_k - \xi_{k+1}} \mathcal{G}_0^L(k, k + 2) + \frac{i}{\xi_{k+1} - \xi_k} \mathcal{G}_0^L(k + 1, k + 2) \right] p(k + 2) , \end{aligned} \quad (5.7)$$

where the correlation function $\mathcal{G}_0^L(k, l) = \langle 0 | C(\xi_k)C(\xi_l) | \mu_1 \mu_2 \rangle$ can be computed using expressions (2.31) and (2.32) for the scalar product. The factor $p(k + 2)$, given by

$$p(l) = \prod_{j=1}^l \frac{\mu_1 - \xi - g\kappa_j}{\mu_1 + \xi - g\kappa_j} \frac{\mu_2 - \xi - g\kappa_j}{\mu_2 + \xi - g\kappa_j} , \quad (5.8)$$

collects the contribution from the momenta.⁶ We can in fact extend rather easily expression (5.7) to the case where the spin operators are located at arbitrary sites, $\langle 0 | \sigma_k^+ \sigma_l^+ | \mu_1 \mu_2 \rangle$. As all factors $(A + D)$ have different arguments, they can be trivially commuted. Therefore the correlation function must be invariant under exchange of the inhomogeneities, except for the factors coming from the correlators $\mathcal{G}_0^L(k, l)$. We find

$$\begin{aligned} \langle 0 | \sigma_k^+ \sigma_l^+ | \mu_1 \mu_2 \rangle &= \langle 0 | C(\xi_k) \prod_{j=k+1}^{l-1} (A + D)(\xi_j)C(\xi_l)B(\mu_1)B(\mu_2) | 0 \rangle p(l) = \\ &= \left[\prod_{j=k+1}^{l-1} \frac{\xi_k - \xi_j + i}{\xi_k - \xi_j} \mathcal{G}_0^L(k, l) + \frac{-i}{\xi_k - \xi_{k+1}} \prod_{j=k+2}^{l-1} \frac{\xi_{k+1} - \xi_j + i}{\xi_{k+1} - \xi_j} \mathcal{G}_0^L(k + 1, l) \right. \\ &\quad \left. + \left(\frac{\xi_{k+2} - \xi_{k+1} + i}{\xi_{k+2} - \xi_{k+1}} \right) \frac{-i}{\xi_k - \xi_{k+2}} \prod_{j=k+3}^{l-1} \frac{\xi_{k+2} - \xi_j + i}{\xi_{k+2} - \xi_j} \mathcal{G}_0^L(k + 2, l) + \dots \right] p(l) , \end{aligned} \quad (5.9)$$

⁶Note that if we take $g \rightarrow 0$ all inhomogeneities become the same, and (5.7) reproduces the limit in equation (4.3).

or using the recursion relations

$$\begin{aligned} \langle 0 | \sigma_k^+ \sigma_l^+ | \mu_1 \mu_2 \rangle = & \left[\prod_{m=k+1}^{l-1} \frac{\xi_k - \xi_m + i}{\xi_k - \xi_m} \mathcal{G}_0^L(k, l) + \right. \\ & \left. + \sum_{m=k+1}^{l-1} \left(\prod_{n=k+1}^{m-1} \frac{\xi_k - \xi_n + i}{\xi_k - \xi_n} \right) \frac{-i}{\xi_k - \xi_m} \left(\prod_{n=k+1}^{l-1} \frac{\xi_m - \xi_n + i}{\xi_m - \xi_n} \right) \mathcal{G}_0^L(m, l) \right] p(l) . \end{aligned} \quad (5.10)$$

A similar discussion holds in the case of higher order correlation functions, involving a larger number of magnons. The analysis of the inhomogeneous case is in fact much less entangled than that of the homogeneous Heisenberg chain (see appendix C for a discussion on the case with three magnons). We will however not present the resulting expressions in here.

6 Concluding remarks

The calculation of form factors and correlation functions of local spin operators in a spin chain can be reduced to the evaluation of inner products of Bethe states. In this paper we have developed a systematic approach to the case of spin operators located at arbitrary sites of the spin chain. We have focused our analysis on the $SU(2)$ sector of $\mathcal{N} = 4$ supersymmetric Yang-Mills at weak-coupling, although as discussed in the appendix the extension to other rank one sectors of the theory is immediate. At one-loop the problem amounts to the calculation of form factors and correlation functions in the Heisenberg spin chain. In the case of form factors reducing the computation to a scalar product is rather straightforward. However, the general case of correlation functions in a homogeneous chain is much more involved, because one needs to face the apparent singular behavior of the algebra of the elements of the monodromy matrix. We have solved this problem by showing that the residue arising each time we commute the operators in the monodromy matrix vanishes. We have also included the extension of our computations to the long-range Bethe ansatz recalling that the system can be described as an inhomogeneous spin chain.

Along our computations, and in order to compare results expected from the CBA with results in the ABA, special care was needed with the normalization of states. We have shown that agreement with results coming from the CBA requires that excitations in the spin chain must be defined using Zamolodchikov-Faddeev operators. An important consequence of this is that states in the ABA pick up an S-matrix factor under the exchange

of two rapidities. This property is crucial if we want for instance Watson's relations [28] to be satisfied by the form factors of the theory. An interesting continuation of our work in this paper would be to understand what other constraints are imposed by the remaining axioms in Smirnov's form factor program [29]. In particular it would be very interesting to understand the behavior under crossing transformations of form factors evaluated using algebraic Bethe ansatz techniques. The extension of Smirnov's program for relativistic integrable theories to worldsheet form factors for $AdS_5 \times S^5$ strings was discussed in [27, 30]. The crossing transformation corresponds to a shift in the rapidity by half the imaginary period of the torus that uniformizes the magnon dispersion relation in the AdS/CFT correspondence [31]. However at one-loop order one of the periods of the rapidity torus becomes infinitely large and thus both periodicity and the crossing transformation become invisible. In order to be able to impose periodicity most likely the dressing phase factor needs to be included. A natural question is therefore the extension of the method that we have developed in this paper to include the dressing phase factor.

Another interesting extension of our work is the analysis of the thermodynamical limit where both the number of magnons is comparable with the number of sites of the spin chain. In this limit the determinant expressions for the scalar product of Bethe states can be expressed as contour integrals. We hope our method can be combined with the semiclassical analysis of contour integrals in [19]-[21].

Acknowledgments

We are grateful to M. Herrero-Valea, V. Martín and M. Montero for discussions. The work of R. H. is supported by MICINN through a Ramón y Cajal contract and grant FPA2011-24568, and by BSCH-UCM through grant GR58/08-910770. J. M. N. wishes to thank the Instituto de Física Teórica UAM-CSIC for kind hospitality during this work.

A General form of \mathcal{F}_n^L

In this appendix we are going to obtain the general expression of the function \mathcal{F}_n^L . All along the calculation the limit $\alpha \rightarrow 0$ will be assumed. Using the first recurrence relation in (4.11) and setting both d and $\frac{\partial d}{\partial \lambda}$ to zero we find

$$\mathcal{F}_n^L = \mathcal{F}_0^L + i\mathcal{D}\mathcal{F}_0^L + i\mathcal{D}\mathcal{F}_1^L + \cdots + i\mathcal{D}\mathcal{F}_{n-1}^L. \quad (\text{A.1})$$

If we assume that $n < L - 1$, the second recurrence equation gives

$$\mathcal{D}\mathcal{F}_n^L = \binom{n}{0}\mathcal{D}\mathcal{F}_0^L + \binom{n}{1}\frac{i\mathcal{D}^2}{2!}\mathcal{F}_0^L + \cdots = \sum_{j=0}^n \binom{n}{j}\frac{i^j\mathcal{D}^{j+1}}{(j+1)!}\mathcal{F}_0^L. \quad (\text{A.2})$$

Therefore we need to sum the series

$$\sum_{j=0}^{n-1} i\mathcal{D}\mathcal{F}_j^L = \sum_{j=0}^{n-1} \sum_{k=0}^j \binom{j}{k} \frac{i^{k+1}\mathcal{D}^{k+1}}{(k+1)!}\mathcal{F}_0^L. \quad (\text{A.3})$$

As a first step, we can commute the two sums as $\sum_{j=0}^{n-1} \sum_{k=0}^j = \sum_{k=1}^{n-1} \sum_{j=k}^{n-1} + \sum_{j=0}^{n-1} \delta_{k,0}$, because the j only appears in the limit of the sum and in the binomial coefficient, so is easy to perform first the sum over j . The second sum is easy to perform because we only have to calculate $\sum_{j=0}^{n-1} \binom{j}{0} = \binom{n}{1}$. The sum over j of the other term can be evaluated using the properties of the binomial coefficients $\sum_{j=k}^{n-1} \binom{j}{k} = \binom{n-1+1}{k+1}$. Then the whole sum can be rewritten as

$$\mathcal{F}_n^L = \mathcal{F}_0^L + \sum_{k=1}^n \binom{n}{k} \frac{i^k\mathcal{D}^k}{k!}\mathcal{F}_0^L = \sum_{k=0}^n \binom{n}{k} \frac{i^k\mathcal{D}^k}{k!}\mathcal{F}_0^L. \quad (\text{A.4})$$

This equation is true $\forall n < L - 1$. If we want to calculate it for $n \geq L - 1$ we have to take into account derivatives of d of order greater or equal to L , which can be done independently of the calculation we have already done, because

$$\mathcal{D}^{L+\alpha-1}\mathcal{F}_{j+1}^L = \frac{i}{L+\alpha}\mathcal{F}_j^L \frac{\partial^{L+\alpha}d}{\partial\lambda^{L+\alpha}} + \cdots,$$

where the dots stand for the part that we have already taken into account. Therefore the d -contribution to $\mathcal{D}\mathcal{F}_M^L$ will be of the form

$$\begin{aligned} i\mathcal{D}\mathcal{F}_M^L &= \sum_{j=1}^{M-L+2} \sum_{k=0}^{M+2-L-j} \frac{i^{L+k-1}}{(L+k-1)!} \binom{M-j}{L+k-2} \mathcal{D}^{L+k-1}\mathcal{F}_j^L \\ &= \sum_{j=1}^{M-L+2} \sum_{k=0}^{M+2-L-j} \frac{i^{L+k}}{(L+k)!} \binom{M-j}{L+k-2} \frac{\partial^{L+k}d}{\partial\lambda^{L+k}} \mathcal{F}_{j-1}^L, \end{aligned}$$

and the derivative of d can be calculated using Leibniz's rule,

$$\left. \frac{\partial^{L+k}d}{\partial\lambda^{L+k}} \right|_{\xi} = \sum_{j=0}^{L+k} \binom{L+k}{j} \frac{\partial^j(\lambda-\xi)^L}{\partial\lambda^j} \frac{\partial^{L+k-j}(\lambda+\xi)^{-L}}{\partial\lambda^{L+k-j}},$$

because we are going to evaluate it at $\lambda = \xi$, the only non-zero contribution is that of L derivatives in the first term, so that $j = L$ and

$$\left. \frac{\partial^{L+k}d}{\partial\lambda^{L+k}} \right|_{\xi} = \binom{L+k}{L} \frac{\partial^L(\lambda-\xi)^L}{\partial\lambda^L} \frac{\partial^k(\lambda+\xi)^{-L}}{\partial\lambda^k} = \frac{(L+k)!}{k!} \frac{(L+k-1)!}{(L-1)!} \frac{(-1)^k}{i^{L+k}}.$$

If we substitute that we obtain

$$i\mathcal{DF}_M^L = \sum_{j=1}^{M-L+2} \sum_{k=0}^{M+2-L-j} \binom{M-j}{L+k-2} \frac{(-1)^k (L+k-1)!}{(L-1)!k!} \mathcal{F}_{j-1}^L .$$

If we perform the sum in k we have

$$\begin{aligned} \sum_{k=0}^{m+2} (-1)^k \binom{L+m}{L+k-2} \binom{L+k-1}{k} &= \frac{(L+m)!}{(L-1)!} \sum_{k=0}^{m+2} \frac{(L+k-1)!}{(L+k-2)!} \frac{(-1)^k}{(m-k+2)!k!} \\ &= \frac{(L+m)!}{(L-1)!(m+2)!} \sum_{k=0}^{m+2} \left[(-1)^k (L-1) \binom{m+2}{k} + (-1)^k k \binom{m+2}{k} \right] , \end{aligned}$$

where $m = M - L - j$. Properties of the binomial coefficients say that the first sum is zero (unless there is a single term, that is, if $m+2=0$) and the second sum is also zero (except if there are two terms, so that $m+2=1$). Then the total contribution of this terms will be

$$\begin{aligned} \sum_{M=L-1}^{n-1} i\mathcal{DF}_M^L &= \sum_{M=L-1}^{n-1} \sum_{j=1}^{M-L+2} \frac{(M-j)!}{(L-1)!(M-L-j+2)!} \\ &\times \left[(L-1)\delta_{M-L-j+2,0} - (M-L-j+2)\delta_{M-L-j+2,1} \right] \mathcal{F}_{j-1}^L , \end{aligned} \quad (\text{A.5})$$

which telescopes, so that

$$\sum_{M=L-1}^{n-1} i\mathcal{DF}_M^L = \mathcal{F}_{n-L}^L . \quad (\text{A.6})$$

Therefore, the most general form of the correlation function \mathcal{F}_n^L is

$$\mathcal{F}_n^L = \sum_{k=0}^n \binom{n}{k} \frac{i^k \mathcal{D}^k}{k!} \mathcal{F}_0^L + \theta(n-L) \mathcal{F}_{n-L}^L . \quad (\text{A.7})$$

where $\theta(x)$ is the Heaviside step function, with $\theta(x) = 0$ if $x < 0$ and $\theta(x) = 1$ if $x \geq 0$.

B Extension to $SL(2)$ and $SU(1|1)$ sectors

Along the main part of this article we have considered the $SU(2)$ spin 1/2 homogeneous Heisenberg spin chain. The whole analysis that we have presented relies on the commutation relations of the elements of the monodromy matrix, equations (2.17), the explicit form of the eigenvalues $a(\lambda)$ and $d(\lambda)$, and the S-matrix (and thus the Bethe ansatz equations). In this appendix we are going to extend these building blocks of the computation to the case of other spin chains with symmetries $SL(2)$ and $SU(1|1)$.

B.1 $SL(2)$ sector

The case of the $SL(2)$ spin chain seems at first sight rather similar to the $SU(2)$ chain, because the commutation relationships between $(A + D)$ and B are the same in both cases. However the eigenvalues a and d are exchanged,

$$a_{SL(2)}(\lambda) = d_{SU(2)}(\lambda) , \quad d_{SL(2)}(\lambda) = a_{SU(2)}(\lambda) = 1 . \quad (\text{B.1})$$

Fortunately, this does not prevent us from repeating the analysis we have developed in section 4 to obtain for instance a set of recurrence equations for correlation functions with two spin operators, like equations (4.11) and (4.25). The derivation is just the same as in that section, but keeping terms in a rather than terms in d . The final result is

$$\begin{aligned} \mathcal{F}_{n+1}^{L,(-1)}(\alpha) &= \left[1 + a(\xi + \alpha) - i \frac{\partial a}{\partial \lambda} \Big|_{\xi+\alpha} \right] \mathcal{F}_n^{L,(-1)}(\alpha) + i[a(\xi + \alpha) - 1] \mathcal{D} \mathcal{F}_n^{L,(-1)}(\alpha) , \\ \mathcal{D}^m \mathcal{F}_{n+1}^{L,(-1)}(\alpha) &= [1 + a(\xi + \alpha)] \mathcal{D}^m \mathcal{F}_n^{L,(-1)}(\alpha) - \frac{i}{m+1} \frac{\partial^{m+1} a}{\partial \alpha^{m+1}} \mathcal{F}_n^{L,(-1)}(\alpha) \\ &\quad + \frac{i}{m+1} [a(\xi + \alpha) - 1] \mathcal{D}^{m+1} \mathcal{F}_n^{L,(-1)}(\alpha) , \\ \mathcal{D}^m \mathcal{F}_0^{L,(-1)}(\alpha) &= \frac{\partial^m \mathcal{F}_0^{L,(-1)}(\alpha)}{\partial \alpha^m} , \end{aligned} \quad (\text{B.2})$$

where the (-1) superindex reminds that now we are calculating the correlation function in an $SL(2)$ spin chain. As in the $SU(2)$ sector,

$$\mathcal{F}_n^{L,(-1)} = \sum_{k=0}^n \binom{n}{k} \frac{(-i)^k \mathcal{D}^k}{k!} \mathcal{F}_0^{L,(-1)} + \theta(n-L) \mathcal{F}_{n-L}^{L,(-1)} . \quad (\text{B.3})$$

B.2 $SU(1|1)$ sector

The case of an $SU(1|1)$ spin chain is slightly more complex to handle because of the grading of the algebra. However the evaluation of correlation functions turns simpler than in the

$SU(2)$ and $SL(2)$ sectors. The commutation relations are given by [32],

$$\begin{aligned}
B(\mu)B(\lambda) &= -\frac{\mu - \lambda + i}{\mu - \lambda - i}B(\lambda)B(\mu) , \\
A(\mu)B(\lambda) &= \left(1 + \frac{i}{\mu - \lambda}\right) B(\lambda)A(\mu) + \frac{i}{\mu - \lambda}B(\mu)A(\lambda) , \\
D(\mu)B(\lambda) &= \left(1 + \frac{i}{\mu - \lambda}\right) B(\lambda)D(\mu) + \frac{i}{\mu - \lambda}B(\mu)D(\lambda) , \\
C(\lambda)A(\mu) &= \left(1 + \frac{i}{\lambda - \mu}\right) A(\mu)C(\lambda) - \frac{i}{\lambda - \mu}A(\lambda)C(\mu) , \\
C(\lambda)D(\mu) &= \left(1 + \frac{i}{\lambda - \mu}\right) D(\mu)C(\lambda) - \frac{i}{\lambda - \mu}D(\lambda)C(\mu) .
\end{aligned} \tag{B.4}$$

These commutation relations present some differences with respect to their $SU(2)$ counterparts. The most important one is that they have the same form both for A and D . Another important difference is that the transfer matrix has to be graded and thus $T(\lambda) = A(\lambda) - D(\lambda)$ instead of $A(\lambda) + D(\lambda)$. On the contrary the form of the functions a and d does not change.

We can now follow section 4 and proceed to find the commutation relations between the transfer matrix and the C operators in the limit where the rapidities are equal. We obtain

$$\begin{aligned}
\lim_{\beta \rightarrow \alpha} C(\alpha)(A - D)(\beta) &= \lim_{\beta \rightarrow \alpha} \frac{\alpha - \beta + i}{\alpha - \beta}(A - D)(\beta)C(\alpha) - \frac{i}{\alpha - \beta}(A - D)(\alpha)C(\beta) \\
&= (A - D)(\alpha)C(\alpha) + i \left[(A - D)(\alpha) \frac{\partial C(\lambda)}{\partial \lambda} \Big|_{\lambda=\alpha} - \frac{\partial (A - D)(\lambda)}{\partial \lambda} \Big|_{\lambda=\alpha} C(\alpha) \right] .
\end{aligned} \tag{B.5}$$

We can also calculate the derivatives and thus the recurrence relations become

$$\begin{aligned}
\mathcal{F}_{n+1}^{L,(0)}(\alpha) &= (1 - d + i\partial d)\mathcal{F}_n^{L,(0)}(\alpha) + i(1 - d)\mathcal{D}\mathcal{F}_n^{L,(0)}(\alpha) , \\
\mathcal{D}^m \mathcal{F}_{n+1}^{L,(0)}(\alpha) &= (1 - d)\mathcal{D}^m \mathcal{F}_n^{L,(0)}(\alpha) \\
&\quad + \frac{i}{m+1} \left[(1 - d)\mathcal{D}^{m+1} \mathcal{F}_n^{L,(0)}(\alpha) + \frac{\partial^{m+1} d}{\partial \alpha^{m+1}} \mathcal{F}_n^{L,(0)}(\alpha) \right] , \\
\mathcal{D}^m \mathcal{F}_0^{L,(0)}(\alpha) &= \frac{\partial^m \mathcal{F}_0^{L,(0)}(\alpha)}{\partial \alpha^m} ,
\end{aligned} \tag{B.6}$$

where the (0) superindex states that now we are calculating a correlation function in the case of an $SU(1|1)$ spin chain, and where as usual $d = d(\xi + \alpha)$ and $\partial d = \frac{\partial d(\lambda)}{\partial \lambda} \Big|_{\lambda=\xi+\alpha}$.

These recurrence equations can again be written in terms of some starting condition corresponding to $n = 0$, using

$$\mathcal{F}_n^{L,(0)} = \sum_{k=0}^n \binom{n}{k} \frac{i^k \mathcal{D}^k}{k!} \mathcal{F}_0^{L,(0)} , \quad (\text{B.7})$$

provided we keep $n < L - 1$.

C Correlation functions involving three magnons

Extracting information from the Bethe equations becomes a challenge when the number of magnons increases. The method that we have developed along this paper can however still applied to evaluate correlations functions involving three magnons. In this appendix we will present a general prescription to calculate these kind of correlators.

There are four non-vanishing correlation functions where the most populated state contains three magnons. The first one is just the scalar product $\langle \lambda_1, \lambda_2, \lambda_3 | \mu_1, \mu_2, \mu_3 \rangle$ and can be directly calculated using the Gaudin formula (2.33). The second one is the form factor of a single spin operator, $\langle \lambda_1, \lambda_2 | \sigma_k^+ | \mu_1, \mu_2, \mu_3 \rangle$, which can be evaluated in a straightforward extension of the computation of $\langle \lambda | \sigma_k^+ | \mu_1 \mu_2 \rangle$ in section 3. The third kind of correlation function involving three magnons is $\langle \lambda | \sigma_k^+ \sigma_l^+ | \mu_1 \mu_2 \mu_3 \rangle$, and the fourth one is $\langle 0 | \sigma_k^+ \sigma_l^+ \sigma_m^+ | \mu_1 \mu_2 \mu_3 \rangle$. These last two types of correlators are the ones that we will consider in this appendix. Actually we will start with the third one, and along the computation we will find that it involves correlation functions of the form $\langle 0 | \sigma_k^+ \sigma_{k+1}^+ \sigma_{k+n+2}^+ | \mu_1 \mu_2 \mu_3 \rangle$, which are a particular case of the fourth type of correlator.

We will start by bringing again the problem to the ABA using relation (2.27),

$$\langle \lambda | \sigma_k^+ \sigma_l^+ | \mu_1 \mu_2 \mu_3 \rangle = \langle 0 | C(\lambda) (A + D)^{k-1}(\xi) C(\xi) (A + D)^n(\xi) C(\xi) (A + D)^{L-l}(\xi) | \mu_1 \mu_2 \mu_3 \rangle , \quad (\text{C.1})$$

where as before $n = L + l - k - 1$. The factor $(A + D)^{k-1}$ acts on $C(\lambda)$ to give $e^{-ip_\lambda(k-1)}$, and the factor $(A + D)^{L-l}$ acts on the three-magnon state to give $e^{-i(p_1+p_2+p_3) \cdot (L-l)} = e^{i(p_1+p_2+p_3)l}$, where in the last equality we have used the periodicity condition for the Bethe roots. Therefore our main problem will be to find the correlation function

$$\mathcal{H}_n^L(\alpha) = \langle 0 | C(\lambda) C(\xi + \alpha) (A + D)^n(\xi) C(\xi) B(\mu_1) B(\mu_2) B(\mu_3) | 0 \rangle . \quad (\text{C.2})$$

Following the procedure that we have constructed along the paper this can be done by relating $\mathcal{H}_{n+1}^L(\alpha)$ to $\mathcal{H}_n^L(\alpha)$. In order to do this let us start by introducing

$$\mathcal{H}_{n+1}^L(\lambda, \alpha, \delta) = \lim_{\beta \rightarrow \alpha} \langle 0 | C(\lambda) C(\xi + \alpha) (A + D)(\xi + \beta) \mathcal{O}(\delta) | 0 \rangle . \quad (\text{C.3})$$

Now we just need to apply the commutation relations (2.17) two times in each step to get

$$\begin{aligned} \mathcal{H}_{n+1}^L(\lambda, \alpha, \delta) &= \lim_{\beta \rightarrow \alpha} \left\{ [1 + d(\xi + \beta)] \mathcal{H}_n^L(\lambda, \alpha, \delta) \right. \\ &\quad - \frac{i}{\lambda - \xi - \beta} [(d(\xi + \beta) - 1) \mathcal{H}_n^L(\lambda, \alpha, \delta) - (d(\lambda) - 1) \mathcal{H}_n^L(\xi + \beta, \alpha, \delta)] \\ &\quad - \frac{i}{\alpha - \beta} [(d(\xi + \beta) - 1) \mathcal{H}_n^L(\lambda, \alpha, \delta) - (d(\xi + \alpha) - 1) \mathcal{H}_n^L(\lambda, \beta, \delta)] \\ &\quad + \frac{i}{\alpha - \beta} \frac{i}{\lambda - \xi - \beta} [(d(\xi + \beta) + 1) \mathcal{H}_n^L(\lambda, \alpha, \delta) - (d(\lambda) + 1) \mathcal{H}_n^L(\xi + \beta, \alpha, \delta)] \\ &\quad \left. - \frac{i}{\alpha - \beta} \frac{i}{\lambda - \xi - \alpha} [(d(\xi + \alpha) + 1) \mathcal{H}_n^L(\lambda, \beta, \delta) - (d(\lambda) + 1) \mathcal{H}_n^L(\xi + \beta, \alpha, \delta)] \right\} . \quad (\text{C.4}) \end{aligned}$$

Taking the limit and applying the Bethe equation for the rapidity λ we obtain

$$\begin{aligned} \mathcal{H}_{n+1}^L(\lambda, \alpha, \delta) &= \left(1 + d + i\partial d + \frac{\partial d - i(d - 1)}{\lambda - \xi - \alpha} + \frac{d + 1}{(\lambda - \xi - \alpha)^2} \right) \mathcal{H}_n^L(\lambda, \alpha, \delta) \\ &\quad + \left[i(1 - d) - \frac{d + 1}{\lambda - \xi - \alpha} \right] \frac{\partial \mathcal{H}_n^L(\lambda, \alpha, \delta)}{\partial \alpha} - \frac{2}{(\lambda - \xi - \alpha)^2} \mathcal{H}_n^L(\xi + \alpha, \alpha, \delta) , \quad (\text{C.5}) \end{aligned}$$

where as before $d = d(\xi + \alpha)$ and $\partial d = \frac{\partial d}{\partial \lambda} \big|_{\xi + \alpha}$. The next step of the calculation is a little bit more involved than in the previous cases because according to (C.5) information about both functions $\mathcal{H}_n^L(\lambda, \alpha, \delta)$ and $\mathcal{H}_{n+1}^L(\xi + \alpha, \alpha, \delta)$ is now needed. This will turn the computation slightly more difficult but still manageable. For convenience in the expressions below we will define $\mathcal{H}_{n+1}^L(\alpha + \xi, \alpha, \delta) = \hat{\mathcal{H}}_{n+1}^L(\alpha, \alpha, \delta)$. This function $\hat{\mathcal{H}}$ has a nice interpretation because

$$\begin{aligned} \langle 0 | \sigma_k^+ \sigma_{k+1}^+ \sigma_{k+n+2}^+ | \mu_1 \mu_2 \mu_3 \rangle &= \langle 0 | C(\xi) C(\xi) (A + D)^n(\xi) C(\xi) (A + D)^{L-n-k-2}(\xi) | \mu_1 \mu_2 \mu_3 \rangle \\ &= \hat{\mathcal{H}}_n^L e^{i(p_1 + p_2 + p_3)(n+k+2)} . \quad (\text{C.6}) \end{aligned}$$

Our starting point is thus to find the recursive equation for $\hat{\mathcal{H}}$. This can be obtained

setting $\lambda = \gamma + \xi$ in expression (C.4) and taking the limit $\gamma \rightarrow \alpha$,

$$\begin{aligned}
\hat{\mathcal{H}}_{n+1}^L(\alpha, \alpha, \delta) &= \lim_{\beta \rightarrow \alpha} \frac{1}{\beta - \alpha} \left[(1+d) \left. \frac{\partial \hat{\mathcal{H}}_n^L(\lambda, \alpha, \delta)}{\partial \lambda} \right|_{\lambda=\alpha} - (1+d) \left. \frac{\partial \hat{\mathcal{H}}_n^L(\alpha, \lambda, \delta)}{\partial \lambda} \right|_{\lambda=\alpha} \right] \\
&+ \left(1+d + 2i\partial d - \frac{1}{2}\partial^2 d \right) \hat{\mathcal{H}}_n^L(\alpha, \alpha, \delta) + [2i(1-d) + \partial d] \left. \frac{\partial \hat{\mathcal{H}}_n^L(\lambda, \alpha, \delta)}{\partial \lambda} \right|_{\lambda=\alpha} \\
&+ \frac{1+d}{2} \left. \frac{\partial^2 \hat{\mathcal{H}}_n^L(\lambda, \alpha, \delta)}{\partial \lambda^2} \right|_{\lambda=\alpha} - (1+d) \left. \frac{\partial^2 \hat{\mathcal{H}}_n^L(\lambda_1, \lambda_2, \delta)}{\partial \lambda_1 \partial \lambda_2} \right|_{\substack{\lambda_1=\alpha \\ \lambda_2=\alpha}}. \tag{C.7}
\end{aligned}$$

Note that although the first term in this expression seems divergent, it vanishes because of the commutation of the C operators, which makes the two derivatives equal. However, this way of calculating recursively $\hat{\mathcal{H}}(\alpha, \alpha, \delta)$ is going to create more problem than it solves, because it will imply calculating the recurrence equation of derivative of $\hat{\mathcal{H}}(\lambda, \mu)$ with respect to either the first or the second argument. Therefore we are going to give the recursion relation of $\hat{\mathcal{H}}(\beta, \alpha, \delta)$ but without taking the limit $\beta \rightarrow \alpha$. To obtain this recurrence relation we only need to substitute $\lambda = \xi + \beta$ in equation (C.5), but *without* imposing $d(\lambda) = 1$,

$$\begin{aligned}
\hat{\mathcal{H}}_{n+1}^L(\beta, \alpha, \delta) &= \left(1+d + i\partial d + \frac{\partial d - i(d-1)}{\beta - \alpha} + \frac{d+1}{(\beta - \alpha)^2} \right) \hat{\mathcal{H}}_n^L(\beta, \alpha, \delta) \\
&+ \left[i(1-d) - \frac{d+1}{\beta - \alpha} \right] \frac{\partial \hat{\mathcal{H}}_n^L(\beta, \alpha, \delta)}{\partial \alpha} + \left[\frac{i(d'-1)}{\beta - \alpha} - \frac{d'+1}{(\beta - \alpha)^2} \right] \lim_{\gamma \rightarrow \alpha} \hat{\mathcal{H}}_n^L(\gamma, \alpha, \delta), \tag{C.8}
\end{aligned}$$

where $d' = d(\xi + \beta)$. Note that if we take $\beta \rightarrow \alpha$ equation (C.8) gives (C.7). Now in the recurrence relation we need to include $\lim_{\gamma \rightarrow \alpha} \hat{\mathcal{H}}_n^L(\gamma, \alpha, \delta)$, but this quantity is obviously known once we know $\hat{\mathcal{H}}_n^L(\beta, \alpha, \delta)$.

We also have need a recurrence equation for the derivatives. For the case of \mathcal{H}_n we

have

$$\begin{aligned}
\mathcal{D}^n \mathcal{H}_{m+1}^L(\lambda, \alpha, \delta) &= (1+d) \mathcal{D}^n \mathcal{H}_m^L - \frac{i}{\lambda - \xi - \alpha} [(d-1) \mathcal{D}^n \mathcal{H}_m^L] \\
&+ \frac{i}{n+1} (\partial^{n+1} d) \mathcal{H}_m^L + (1-d) \frac{i}{n+1} \mathcal{D}^{n+1} \mathcal{H}_m^L \\
&+ \sum_{k=0}^n \sum_{l=0}^{k+1} \frac{n!}{(n-k)!(k+1-l)!} \frac{1}{(\lambda - \xi - \alpha)^{l+1}} [\partial^{k+1-l} (d+1) \mathcal{D}^{n-k} \mathcal{H}_m^L \\
&- (d(\lambda) + 1) \mathcal{D}_1^{n-k} \mathcal{D}_2^{k+1-l} \hat{\mathcal{H}}_m^L] \\
&- \sum_{k=0}^n \sum_{l=0}^{n-k} \frac{n!}{(k+1)!(n-l-k)!} \frac{1}{(\lambda - \xi - \alpha)^{l+1}} [\partial^{n-k-l} (d+1) \mathcal{D}^{k+1} \mathcal{H}_m^L \\
&- (d(\lambda) + 1) \mathcal{D}_1^{n-k-l} \mathcal{D}_2^{k+1} \hat{\mathcal{H}}_m^L] . \tag{C.9}
\end{aligned}$$

The last two sums cancel themselves except for the terms with $k = n$. Therefore

$$\begin{aligned}
\mathcal{D}^n \mathcal{H}_{m+1}^L(\lambda, \alpha, \delta) &= (1+d) \mathcal{D}^n \mathcal{H}_m^L - \frac{i}{\lambda - \xi - \alpha} (d-1) \mathcal{D}^n \mathcal{H}_m^L \\
&+ \frac{i}{n+1} (\partial^{n+1} d) \mathcal{H}_m^L + (1-d) \frac{i}{n+1} \mathcal{D}^{n+1} \mathcal{H}_m^L \\
&+ \sum_{l=0}^{n+1} \frac{n!}{(n+1-l)!} \frac{1}{(\lambda - \xi - \alpha)^{l+1}} [\partial^{n+1-l} (d+1) \mathcal{H}_m^L - 2 \mathcal{D}^{n+1-l} \hat{\mathcal{H}}_m^L] \\
&- \frac{1}{n+1} \frac{1}{(\lambda - \xi - \alpha)} [(d+1) \mathcal{D}^{n+1} \mathcal{H}_m^L - 2 \mathcal{D}^{n+1} \hat{\mathcal{H}}_m^L] , \tag{C.10}
\end{aligned}$$

where we have used that $d(\lambda) = 1$. In a similar way we can obtain an expression for the derivatives of $\hat{\mathcal{H}}$,

$$\begin{aligned}
\mathcal{D}^n \hat{\mathcal{H}}_{m+1}^L(\beta, \alpha, \delta) &= (1+d) \mathcal{D}^n \hat{\mathcal{H}}_m^L - \frac{i}{\beta - \alpha} (d-1) \mathcal{D}^n \hat{\mathcal{H}}_m^L \\
&+ \frac{i}{n+1} (\partial^{n+1} d) \hat{\mathcal{H}}_m^L + (1-d) \frac{i}{n+1} \mathcal{D}^{n+1} \hat{\mathcal{H}}_m^L + \frac{i}{\beta - \alpha} (d' - 1) \lim_{\gamma \rightarrow \alpha} \frac{\partial^n \hat{\mathcal{H}}_n^L(\gamma, \alpha, \delta)}{\partial \alpha^n} \\
&+ \sum_{l=0}^{n+1} \frac{n!}{(n+1-l)!} \frac{1}{(\beta - \alpha)^{l+1}} \left[\partial^{n+1-l} (d+1) \hat{\mathcal{H}}_m^L - (d' + 1) \lim_{\gamma \rightarrow \alpha} \frac{\partial^{n+1-l} \hat{\mathcal{H}}_m^L(\gamma, \alpha, \delta)}{\partial \alpha^{n+1-l}} \right] \\
&- \frac{1}{n+1} \frac{1}{(\beta - \alpha)} \left[(d+1) \mathcal{D}^{n+1} \hat{\mathcal{H}}_m^L - (d' + 1) \lim_{\gamma \rightarrow \alpha} \frac{\partial^{n+1} \hat{\mathcal{H}}_m^L(\gamma, \alpha, \delta)}{\partial \alpha^{n+1}} \right] . \tag{C.11}
\end{aligned}$$

At this point the problem is, at least formally, solved. We have found the recursion relation for $\hat{\mathcal{H}}$ and its derivatives, with $\langle 0 | C(\xi + \beta) C(\xi + \alpha) (\xi) C(\xi) | \mu_1 \mu_2 \mu_3 \rangle = \hat{\mathcal{H}}_n^L(\beta, \alpha)$

as the initial condition. These functions can then be substituted in the recursion relation for \mathcal{H} and thus we can obtain the desired correlation function. However, we are not going to present the general form for the correlation function \mathcal{H}_n^L as function of \mathcal{H}_0^L , $\hat{\mathcal{H}}_0^L$ and their derivatives, because although straightforward it becomes rather lengthy. This is because when we substitute the expression for the derivatives the recursion relations turn to depend on all the \mathcal{H}_i with $0 \leq i \leq n$, even once we take the limit $\alpha \rightarrow 0$. Instead we can present the case of correlation functions with n small, to exhibit the nested procedure to write the result in terms of the initial functions \mathcal{H}_0^L and $\hat{\mathcal{H}}_0^L$. In particular we are going to consider the first three functions, with $n = 1$, $n = 2$ and $n = 3$. Thus we can safely assume that $n < L - 1$ so that all the d and $\partial^k d$ factors can be set to zero in the limit $\alpha \rightarrow 0$. The first of these correlation functions is given by

$$\mathcal{H}_1^L = (1 + ic(\lambda) + c(\lambda)^2) \mathcal{H}_0^L + (i - c(\lambda)) \left. \frac{\partial \mathcal{H}_0^L(\lambda, \alpha)}{\partial \alpha} \right|_{\alpha=0} - 2c(\lambda)^2 \hat{\mathcal{H}}_0^L, \quad (\text{C.12})$$

where for convenience we have defined $c(\lambda) = 1/(\lambda - \xi)$. For simplicity, if no arguments of this functions are given, $\mathcal{H}^L(\lambda, 0)$ and $\hat{\mathcal{H}}^L(0, 0)$ must be understood. The last step of the computation reduces to calculating some initial conditions, which now are

$$\mathcal{H}_0^L(\lambda, \alpha) = \langle 0 | C(\lambda) C(\xi + \alpha) C(\xi) B(\mu_1) B(\mu_2) B(\mu_3) | 0 \rangle, \quad (\text{C.13})$$

$$\hat{\mathcal{H}}_0^L(\alpha, \beta) = \mathcal{H}_0^L(\xi + \alpha, \beta). \quad (\text{C.14})$$

These functions can be easily computed using equations (2.32). However we are not going to present the explicit expression for these scalar products because of its length and because we want to show the way to solve the recurrence relation rather than obtaining the explicit value of the correlation function.

The functional dependence of \mathcal{H}_1 on \mathcal{H}_0 is repeated for a given value of n and the lower correlator. That is, in the limit $\alpha \rightarrow 0$ the recurrence relation for \mathcal{H}_{n+1}^L is given by

$$\mathcal{H}_{n+1}^L = (1 + ic(\lambda) + c(\lambda)^2) \mathcal{H}_n^L + (i - c(\lambda)) \mathcal{D} \mathcal{H}_n^L - 2c(\lambda)^2 \hat{\mathcal{H}}_n. \quad (\text{C.15})$$

Therefore for the second correlation function we have

$$\mathcal{H}_2^L = (1 + ic(\lambda) + c(\lambda)^2) \mathcal{H}_1^L + (i - c(\lambda)) \mathcal{D} \mathcal{H}_1^L - 2c(\lambda)^2 \hat{\mathcal{H}}_1. \quad (\text{C.16})$$

As we already know \mathcal{H}_1^L it only remains to find the other two functions entering (C.16).

This can be done using the equations that we have obtained in this appendix. We get

$$\begin{aligned} \mathcal{D}\mathcal{H}_1^L &= c(\lambda)^3 \mathcal{H}_0^L + (1 + ic(\lambda)) \left. \frac{\partial \mathcal{H}_0^L(\lambda, \alpha)}{\partial \alpha} \right|_{\alpha=0} + \frac{i(1 + ic(\lambda))}{2} \left. \frac{\partial^2 \mathcal{H}_0^L(\lambda, \alpha)}{\partial \alpha^2} \right|_{\alpha=0} \\ &\quad - 2c(\lambda)^3 \hat{\mathcal{H}}_0^L - 2c(\lambda)^2 \left. \frac{\partial \hat{\mathcal{H}}_0^L(0, \alpha)}{\partial \alpha} \right|_{\alpha=0}, \end{aligned} \quad (\text{C.17})$$

$$\hat{\mathcal{H}}_1^L(\beta, 0) = \left(1 + \frac{i}{\beta} + \frac{1}{\beta^2}\right) \hat{\mathcal{H}}_0^L(\beta, 0) + \left[i - \frac{1}{\beta}\right] \left. \frac{\partial \hat{\mathcal{H}}_0^L(\beta, \alpha)}{\partial \alpha} \right|_{\alpha=0} - \left[\frac{i}{\beta} + \frac{1}{\beta^2}\right] \hat{\mathcal{H}}_0^L, \quad (\text{C.18})$$

$$\hat{\mathcal{H}}_1^L = \hat{\mathcal{H}}_0^L + 2i \left. \frac{\partial \hat{\mathcal{H}}_0^L(0, \alpha)}{\partial \alpha} \right|_{\alpha=0} + \frac{1}{2} \left. \frac{\partial^2 \hat{\mathcal{H}}_0^L(0, \alpha)}{\partial \alpha^2} \right|_{\alpha=0} - \left. \frac{\partial^2 \hat{\mathcal{H}}_0^L(\alpha, \beta)}{\partial \alpha \partial \beta} \right|_{\substack{\alpha=0 \\ \beta=0}}, \quad (\text{C.19})$$

which reduce again to some dependence on the initial conditions we have described before.

An identical computation can be done for \mathcal{H}_3^L ,

$$\mathcal{H}_3^L = (1 + ic(\lambda) + c(\lambda)^2) \mathcal{H}_2^L + (i - c(\lambda)) \mathcal{D}\mathcal{H}_2^L - 2c(\lambda)^2 \hat{\mathcal{H}}_2. \quad (\text{C.20})$$

Now, besides \mathcal{H}_2^L , that has been calculated just before, we need

$$\begin{aligned} \mathcal{D}\mathcal{H}_2^L &= c(\lambda)^3 \mathcal{H}_1^L + (1 + ic(\lambda)) \mathcal{D}\mathcal{H}_1^L + \frac{i(1 + ic(\lambda))}{2} \mathcal{D}^2 \mathcal{H}_1^L \\ &\quad - 2c(\lambda)^2 \left(c(\lambda) \hat{\mathcal{H}}_1^L + \mathcal{D}\hat{\mathcal{H}}_1^L \right), \end{aligned} \quad (\text{C.21})$$

$$\begin{aligned} \mathcal{D}^2 \mathcal{H}_1^L &= 2c(\lambda)^4 \mathcal{H}_0^L + (1 + ic(\lambda)) \left. \frac{\partial^2 \mathcal{H}_0^L(\lambda, \alpha)}{\partial \alpha^2} \right|_{\alpha=0} + \frac{i(1 + ic(\lambda))}{3} \left. \frac{\partial^3 \mathcal{H}_0^L(\lambda, \alpha)}{\partial \alpha^3} \right|_{\alpha=0} \\ &\quad - 4c(\lambda)^4 \hat{\mathcal{H}}_0^L - 4c(\lambda)^3 \left. \frac{\partial \hat{\mathcal{H}}_0^L(0, \alpha)}{\partial \alpha} \right|_{\alpha=0} - 2c(\lambda)^2 \left. \frac{\partial^2 \hat{\mathcal{H}}_0^L(0, \alpha)}{\partial \alpha^2} \right|_{\alpha=0}, \end{aligned} \quad (\text{C.22})$$

$$\begin{aligned} \mathcal{D}\hat{\mathcal{H}}_1^L &= \left. \frac{\partial \hat{\mathcal{H}}_0^L(0, \alpha)}{\partial \alpha} \right|_{\alpha=0} + \frac{i}{2} \left. \frac{\partial^2 \hat{\mathcal{H}}_0^L(0, \alpha)}{\partial \alpha^2} \right|_{\alpha=0} + i \left. \frac{\partial^2 \hat{\mathcal{H}}_0^L(\alpha, \beta)}{\partial \alpha \partial \beta} \right|_{\substack{\alpha=0 \\ \beta=0}} \\ &\quad + \frac{1}{3!} \left. \frac{\partial^3 \hat{\mathcal{H}}_0^L(0, \alpha)}{\partial \alpha^3} \right|_{\alpha=0} - \frac{1}{2} \left. \frac{\partial^3 \hat{\mathcal{H}}_0^L(\alpha, \beta)}{\partial \alpha \partial \beta^2} \right|_{\substack{\alpha=0 \\ \beta=0}}, \end{aligned} \quad (\text{C.23})$$

$$\begin{aligned} \hat{\mathcal{H}}_2^L &= \hat{\mathcal{H}}_0^L + 4i \left. \frac{\partial \hat{\mathcal{H}}_0^L(0, \alpha)}{\partial \alpha} \right|_{\alpha=0} - 4 \left. \frac{\partial^2 \hat{\mathcal{H}}_0^L(\alpha, \beta)}{\partial \alpha \partial \beta} \right|_{\substack{\alpha=0 \\ \beta=0}} + \frac{i}{2} \left. \frac{\partial^3 \hat{\mathcal{H}}_0^L(0, \alpha)}{\partial \alpha^3} \right|_{\alpha=0} \\ &\quad - \frac{3i}{2} \left. \frac{\partial^3 \hat{\mathcal{H}}_0^L(\alpha, \beta)}{\partial \alpha \partial \beta^2} \right|_{\substack{\alpha=0 \\ \beta=0}} - \frac{1}{3!} \left. \frac{\partial^4 \hat{\mathcal{H}}_0^L(\alpha, \beta)}{\partial \alpha \partial \beta^3} \right|_{\substack{\alpha=0 \\ \beta=0}} + \frac{1}{2!^2} \left. \frac{\partial^4 \hat{\mathcal{H}}_0^L(\alpha, \beta)}{\partial \alpha^2 \partial \beta^2} \right|_{\substack{\alpha=0 \\ \beta=0}}. \end{aligned} \quad (\text{C.24})$$

The cases with higher values of n can be obtained along similar lines.

To conclude our analysis we will brief comment on the calculation of correlation functions $\langle 0 | \sigma_k^+ \sigma_l^+ \sigma_m^+ | \{\mu\} \rangle$, with general values of k , l and m . In this case the value of n in $\hat{\mathcal{H}}_n^L$ will be proportional to the separation of l and m . But still remains to separate k from l . This last step can be solved using the tools from section 4.2, because the problem in both cases is the same.

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